Holonomy groups of Lorentzian manifolds - a status report

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Abstract

In this survey we review the state of art in Lorentzian holonomy theory. We explain the recently completed classification of connected Lorentzian holonomy groups, we describe local and global metrics with special Lorentzian holonomy and some topological properties, and we discuss the holonomy groups of Lorentzian manifolds with parallel spinors as well as Lorentzian Einstein metrics with special holonomy.

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1 Introduction

The holonomy group of a semi-Riemannian manifold (M,g) is the group of all parallel displacements along curves which are closed in a fixed point $x \in M$. This group is a Lie

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subgroup of the group of orthogonal transformations of (T_xM, g_x). The concept of holonomy group was probably first successfully applied in differential geometry by E. Cartan [C25], [C26a], [C26b], who used it to classify symmetric spaces. Since then, it has proved to be a very important concept. In particular, it allows to describe parallel sections in geometric vector bundles associated to (M, g) as holonomy invariant objects and therefore by purely algebraic tools. Moreover, geometric properties like curvature properties can be read off if the holonomy group is special, i.e., a proper subgroup of $\mathrm{O}(T_xM, g_x)$. One of the important consequences of the holonomy notion is its application to the 'classification' of special geometries that are compatible with Riemannian geometry. For each of these geometries an own branch of differential geometry has developed, for example Kähler geometry (holonomy $\mathrm{U}(n)$), geometry of Calabi-Yau manifolds ($\mathrm{SU}(n)$), hyper-Kähler geometry ($\mathrm{Sp}(n)$), quaternionic Kähler geometry ($\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$), geometry of G_2-manifolds or of $\mathrm{Spin}(7)$-manifolds. In physics there is much interest in semi-Riemannian manifolds with special holonomy, since they often allow to construct spaces with additional supersymmetries (Killing spinors). The development of holonomy theory has a long history. We refer for details to [Bry96], [Bry99] and [Be87].

Whereas the holonomy groups of simply connected Riemannian manifolds are completely known since the 50th of the last century, the classification of holonomy groups for pseudo-Riemannian manifolds is widely open, only the irreducible holonomy representations of simply connected pseudo-Riemannian manifolds are known ([Ber55], [Ber57]). The difficulty in case of indefinite metrics is the appearance of degenerate holonomy invariant subspaces. Such holonomy representations are hard to classify.

The holonomy groups of 4-dimensional Lorentzian manifolds were classified by physicists working in General Relativity ([Sch60], [Sh70]), the general dimension was long time ignored. Due to the development of supergravity and string theory in the last decades physicists as well as mathematicians became more interested in higher dimensional Lorentzian geometry. The search for special supersymmetries required the classification of holonomy groups in higher dimension. In the beginning of the 90th, L. Berard-Bergery and his students began a systematic study of Lorentzian holonomy groups. They discovered many special features of Lorentzian holonomy. Their groundbreaking paper [BI93] on the algebraic structure of subgroups $H \subset \mathrm{SO}(1, n-1)$ acting with a degenerate invariant subspace was the starting point for the classification. T. Leistner ([L03], [L07]) completed the classification of the (connected) Lorentzian holonomy groups by the full description of the structure of such $H \subset \mathrm{SO}(1, n-1)$ which can appear as holonomy groups. It remained to show that any of the groups in Leistner's holonomy list can be realized by a Lorentzian metric. Many realizations were known before but some cases were still open until A. Galaev [G05] finally found a realization for all of the groups.

The aim of this review is to describe these classification results and the state of art in Lorentzian holonomy theory. In section 2 we give a short basic introduction to holonomy theory and recall the classification of Riemannian holonomy groups. In section 3 the algebraic classification of (connected) Lorentzian holonomy groups is explained. Section 4 is devoted to the realization of the Lorentzian holonomy groups by local metrics, which completes the classification of these groups. Section 5 deals with global aspects of Lorentzian holonomy theory. First we describe Lorentzian symmetric spaces and their holonomy groups. After that we discuss the holonomy group of Lorentzian cones and a construction of Lorentzian metrics with special holonomy on non-trivial torus bundles. Furthermore, we describe a class of examples of geodesically complete resp. globally hyperbolic Lorentzian manifolds with special holonomy. We close the section with some results on topological properties of Lorentzian manifolds with special holonomy. In section 6 we consider the relation between holonomy groups and parallel spinors, derive the Lorentzian holonomy groups which allow parallel spinors and discuss a construction of globally hyperbolic Lorentzian manifolds with complete Cauchy surfaces and parallel spinors. The final part deals with Lorentzian Einstein manifolds with special holonomy. We describe their holonomy groups and the local structure of the metrics.

2 Holonomy groups of semi-Riemannian manifolds

Let (M,g) be a connected[1] n-dimensional manifold with a metric g of signature[2] (p,q), and let ∇^g be the Levi-Civita connection of (M,g). If $\gamma : [a,b] \to M$ is a piecewise smooth curve connecting two points x and y of M, then for any $v \in T_xM$ there is a uniquely determined parallel vector field X_v along γ with initial value v:

$$\frac{\nabla^g X_v}{dt}(t) = 0 \quad \forall\, t \in [a,b], \quad X_v(a) = v.$$

Since the Levi-Civita connection is metric, the parallel displacement

$$\begin{array}{cccc} \mathscr{P}^g_\gamma : & T_xM & \longrightarrow & T_yM \\ & v & \longmapsto & X_v(b) \end{array}$$

defined by X_v is a linear isometry between (T_xM, g_x) and (T_yM, g_y). In particular, if γ is closed, $\mathscr{P}^g_\gamma$ is an orthogonal map on (T_xM, g_x). The *holonomy group of* (M,g) *with respect to* $x \in M$ is the Lie group

$$\mathrm{Hol}_x(M,g) := \{\, \mathscr{P}^g_\gamma : T_xM \to T_xM \mid \gamma \in \Omega(x) \,\} \subset \mathrm{O}(T_xM, g_x),$$

where $\Omega(x)$ denotes the set of piecewise smooth curves closed in x. If we restrict ourself to null homotopic curves, we obtain the *reduced holonomy group of* (M,g) *with respect to* $x \in M$:

$$\mathrm{Hol}^0_x(M,g) := \{\, \mathscr{P}^g_\gamma : T_xM \to T_xM \mid \gamma \in \Omega(x) \text{ null homotopic } \} \subset \mathrm{Hol}_x(M,g).$$

$\mathrm{Hol}^0_x(M,g)$ is the connected component of the Identity in the Lie group $\mathrm{Hol}_x(M,g)$. Hence, the holonomy group of a simply connected manifold is connected. The holonomy groups of two different points are conjugated: If σ is a smooth curve connecting x with y, then

$$\mathrm{Hol}_y(M,g) = \mathscr{P}^g_\sigma \circ \mathrm{Hol}_x(M,g) \circ \mathscr{P}^g_{\sigma^-}.$$

Therefore, we often omit the reference point and consider the holonomy groups of (M,g) as class of conjugated subgroups of the (pseudo)orthogonal group $\mathrm{O}(p,q)$ (fixing an orthonormal basis in (T_xM, g_x)).
If $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ is the universal semi-Riemannian covering, then

$$\mathrm{Hol}^0_{\tilde{x}}(\widetilde{M}, \widetilde{g}) = \mathrm{Hol}_{\tilde{x}}(\widetilde{M}, \widetilde{g}) \simeq \mathrm{Hol}^0_{\pi(\tilde{x})}(M,g).$$

[1] In this paper all manifolds are supposed to be connected.

[2] In this paper p denotes the number of -1 and q the number of $+1$ in the normal form of the metric. We call (M,g) *Riemannian* if $p = 0$, *Lorentzian* if $p = 1 < n$ and *pseudo-Riemannian* if $1 \le p < n$. If we do not want to specify the signature we say *semi-Riemannian manifold*.

For a semi-Riemannian product $(M,g) = (M_1,g_1) \times (M_2,g_2)$ and $(x_1,x_2) \in M_1 \times M_2$, the holonomy group is the product of its factors

$$\mathrm{Hol}_{(x_1,x_2)}(M,g) = \mathrm{Hol}_{x_1}(M_1,g_1) \times \mathrm{Hol}_{x_2}(M_2,g_2).$$

An important result, which relates the holonomy group to the curvature of (M,g), is the Holonomy Theorem of Ambrose and Singer. We denote by R^g the curvature tensor of (M,g). Due to the symmetry properties of the curvature tensor, for all $x \in M$ and $v,w \in T_xM$ the endomorphism $R^g_x(v,w) : T_xM \to T_xM$ is skew-symmetric with respect to g_x, hence an element of the Lie algebra $\mathfrak{so}(T_xM,g_x)$ of $\mathrm{O}(T_xM,g_x)$. Let γ be a piecewise smooth curve from x to y and $v,w \in T_xM$. We denote by $(\gamma^*R^g)_x(v,w)$ the endomorphism

$$(\gamma^*R^g)_x(v,w) := \mathscr{P}^g_{\gamma^-} \circ R^g_y\big(\mathscr{P}^g_\gamma(v), \mathscr{P}^g_\gamma(w)\big) \circ \mathscr{P}_\gamma \in \mathfrak{so}(T_xM,g_x).$$

The Lie algebra of the holonomy group is generated by the curvature operators $(\gamma^*R^g)_x$, more exactly, the *Holonomy Theorem of Ambrose and Singer* states:

Theorem 2.1 (Holonomy Theorem of Ambrose and Singer) *The Lie algebra of the holonomy group* $\mathrm{Hol}_x(M,g)$ *is given by*

$$\mathfrak{hol}_x(M,g) = \mathrm{span}\left\{ (\gamma^*R^g)_x(v,w) \;\middle|\; \begin{array}{l} v,w \in T_xM, \\ \gamma \textit{ curve with initial point } x \end{array} \right\}.$$

This Theorem provides a tool for the calculation of the holonomy algebra of a manifold which determines the connected component $\mathrm{Hol}^0_x(M,g)$ of the holonomy group. Therefore it is the starting point in the classification of holonomy groups. It is enough to describe holonomy groups of simply connected manifolds, whereas the description of the full holonomy group in the general case is a much more complicated problem.

Another important property of holonomy groups is stated in the following *holonomy principle*, which relates parallel tensor fields on M to fixed elements under the action of the holonomy group.

Theorem 2.2 (Holonomy Principle) *Let $\mathscr{T}$ be a tensor bundle on (M,g) and let ∇^g be the covariant derivative on $\mathscr{T}$ induced by the Levi-Civita connection. If $T \in \Gamma(\mathscr{T})$ is a tensor field with $\nabla^g T = 0$, then $\mathrm{Hol}_x(M,g)\,T(x) = T(x)$, where $\mathrm{Hol}_x(M,g)$ acts in the canonical way on the tensors $\mathscr{T}_x$. Contrary, if $T_x \in \mathscr{T}_x$ is a tensor with $\mathrm{Hol}_x(M,g)\,T_x = T_x$, then there is an uniquely determined tensor field $T \in \Gamma(\mathscr{T})$ with $\nabla^g T = 0$ and $T(x) = T_x$. T is given by parallel displacement of T_x, i.e., $T(y) := \mathscr{P}^{\nabla^g}_\gamma(T_x)$, where $y \in M$ and γ is a curve connecting x with y.*

The holonomy group $\mathrm{Hol}_x(M,g)$ acts as group of orthogonal mappings on the tangent space (T_xM,g_x). This representation is called the *holonomy representation of* (M,g), we denote it in the following by ρ. The holonomy representation $\rho : \mathrm{Hol}_x(M,g) \to \mathrm{O}(T_xM,g_x)$ is called *irreducible* if there is no proper holonomy invariant subspace $E \subset T_xM$. ρ is called *weakly-irreducible*, if there is no proper *non-degenerate* holonomy invariant subspace $E \subset T_xM$. To be short, we say that the holonomy group resp. its Lie algebra is *irreducible (weakly irreducible)*, if the holonomy representation has this property. If (M,g) is a Riemannian manifold, any weakly irreducible holonomy representation is irreducible. In the pseudo-Riemannian case there are weakly irreducible holonomy representations which admit degenerate holonomy invariant subspaces, i.e., which are not irreducible. This causes

the problems in the classification of the holonomy groups of pseudo-Riemannian manifolds.
For a subspace $E \subset T_xM$ we denote by

$$E^{\perp} = \{v \in T_xM \mid g_x(v,E) = 0\} \subset T_xM$$

its orthogonal complement. If E is holonomy invariant, then $E^{\perp}$ is holonomy invariant as well. If E is in addition non-degenerate, then $E^{\perp}$ is non-degenerate as well and T_xM is the direct sum of these holonomy invariant subspaces:

$$T_xM = E \oplus E^{\perp}.$$

For that reason we call weakly-irreducible representations also *indecomposable* (meaning, that they do not decompose into the direct sum of non-degenerate subrepresentations). A semi-Riemannian manifold (M,g) is called *irreducible* if its holonomy representation is irreducible. (M,g) is called *weakly-irreducible* or *indecomposable* if its holonomy representation is weakly-irreducible.

If the holonomy representation of (M,g) has a proper non-degenerate holonomy invariant subspace, then the reduced holonomy group decomposes into a product of groups. Moreover, the manifold itself spits locally into a semi-Riemannian product. More exactly:

Theorem 2.3 (Local Decomposition Theorem) *Let $E \subset T_xM$ be a k-dimensional proper non-degenerate holonomy invariant subspace, then the groups*

$$\begin{array}{rcll} H_1 & := & \{ \mathscr{P}^g_\gamma \in \mathrm{Hol}^0_x(M,g) \mid (\mathscr{P}^g_\gamma)_{|E^\perp} = \mathrm{Id}_{E^\perp} \} & \quad \textit{and} \\ H_2 & := & \{ \mathscr{P}^g_\gamma \in \mathrm{Hol}^0_x(M,g) \mid (\mathscr{P}^g_\gamma)_{|E} = \mathrm{Id}_E \} & \end{array}$$

are normal subgroups of $\mathrm{Hol}^0_x(M,g)$ *and*

$$\mathrm{Hol}^0_x(M,g) \simeq H_1 \times H_2.$$

Moreover, (M,g) is locally isometric to a semi-Riemannian product, i.e., for any point $p \in M$ there exists a neighborhood $U(p)$ and two semi-Riemannian manifolds (U_1,g_1) and (U_2,g_2) of dimension k and $(n-k)$, respectively, such that

$$(U(p),g) \overset{\textit{isometric}}{\simeq} (U_1,g_1) \times (U_2,g_2).$$

The local decomposition of (M,g) follows from the Frobenius Theorem. If $E \subset T_xM$ is a non-degenerate, holonomy invariant subspace, then

$$\mathscr{E} : y \in M \longrightarrow \mathscr{E}_y := \mathscr{P}^g_\sigma(E) \subset T_yM,$$

where σ is a piecewise smooth curve from x to y, is an involutive distribution on M, the *holonomy distribution defined by E*. The maximal connected integral manifolds of $\mathscr{E}$ are totally geodesic submanifolds of M, which are geodesically complete if (M,g) is so. The manifolds (U_1,g_1) and (U_2,g_2) in Theorem 2.3 can be chosen as small open neighborhood of p in the integral manifold $M_1(p)$ of the holonomy distribution $\mathscr{E}$ defined by E and the integral manifold $M_2(p)$ of the holonomy distribution $\mathscr{E}^{\perp}$ defined by $E^{\perp}$, respectively, with the metric induced by g. If (M,g) is simply connected and geodesically complete, (M,g) is even globally isometric to the product of the two integral manifolds $(M_1(p),g_1)$

and $(M_2(p), g_2)$. Now, we decompose the tangent space T_xM into a direct sum of non-degenerate, orthogonal and holonomy invariant subspaces

$$T_xM = E_0 \oplus E_1 \oplus \ldots \oplus E_r,$$

where $\mathrm{Hol}_x(M,g)$ acts weakly irreducible on $E_1, \ldots, E_r$ and E_0 is a maximal subspace (possibly 0-dimensional), on which the holonomy group $\mathrm{Hol}_x(M,g)$ acts trivial. Applying the global version of Theorem 2.3 to this decomposition we obtain the *Decomposition Theorem of de Rham and Wu* ([DR52], [Wu64]).

Theorem 2.4 (Decomposition Theorem of de Rham und Wu) *Let (M,g) be a simply connected, geodesically complete semi-Riemannian manifold. Then (M,g) is isometric to a product of simply connected, geodesically complete semi-Riemannian manifolds*

$$(M,g) \simeq (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_r, g_r),$$

where (M_0, g_0) is a (possibly null-dimensional) (pseudo-)Euclidian space and the factors $(M_1, g_1), \ldots, (M_r, g_r)$ are indecomposable and non-flat.

Theorem 2.4 reduces the classification of reduced holonomy groups of geodesically complete semi-Riemannian manifolds to the study of weakly irreducible holonomy representations. This classification is widely open, but the subcase of *irreducible* holonomy representations is completely solved. First of all, let us mention that the holonomy group of a symmetric space is given by its isotropy representation.

Theorem 2.5 *Let (M,g) be a symmetric space, and let $G(M) \subset \mathrm{Isom}(M,g)$ be its transvection group. Furthermore, let $\lambda : H(M) \longrightarrow \mathrm{GL}(T_{x_0}M)$ be the isotropy representation of the stabiliser $H(M) = G(M)_{x_0}$ of a point $x_0 \in M$. Then,*

$$\lambda(H(M)) = \mathrm{Hol}_{x_0}(M,g).$$

In particular, the holonomy group $\mathrm{Hol}_{x_0}(M,g)$ is isomorphic to the stabilizer $H(M)$ and, using this isomorphism, the holonomy representation ρ is given by the isotropy representation λ.

Therefore, the holonomy groups of symmetric spaces can be read off from the classification lists of symmetric spaces, which describe the pair $(G(M), H(M))$ and the isotropy representation λ. For *irreducible* symmetric spaces these lists can be found in [Be87], chapter 10, in [He01] and in [Ber57]. In order to classify the *irreducible* holonomy representations, the classification of the non-symmetric case remains. This was done by M. Berger in 1955 (cf. [Ber55]. He proved that there is only a short list of groups which can appear as holonomy groups of *irreducible* non-locally symmetric simply connected semi-Riemannian manifolds. This list is now called the *Berger list*. The Berger list of Riemannian manifolds is widely known. There appear only 6 special holonomy groups and due to the holonomy principle (cf. Theorem 2.2) each of these groups is related to a special, rich and interesting geometry, described by the corresponding parallel geometric object.

Theorem 2.6 (Riemannian Berger list) *Let (M^n, g) be an n-dimensional, simply connected, irreducible, non-locally symmetric Riemannian manifold. Then the holomomy* *group* $\mathrm{Hol}(M,g)$ *is up to conjugation in* $\mathrm{O}(n)$ *either* $\mathrm{SO}(n)$ *or one of the following groups with its standard representation:*

n	*holonomy group*	*special geometry*
$2m \geq 4$	$\mathrm{U}(m)$	*Kähler manifold*
$2m \geq 4$	$\mathrm{SU}(m)$	*Ricci-flat Kähler manifold*
$4m \geq 8$	$\mathrm{Sp}(m)$	*Hyperkähler manifold*
$4m \geq 8$	$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$	*quaternionic Kähler manifold*
7	G_2	G_2*-manifold*
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$*-manifold*

The Berger list for pseudo-Riemannian manifolds is given in the following Theorem.

Theorem 2.7 (Pseudo-Riemannian Berger list) *Let* (M,g) *be a simply connected, irreducible, non-locally symmetric semi-Riemannian manifold of signatur* (p,q)*. Then the holonomy group of* (M,g) *is up to conjugation in* $\mathrm{O}(p,q)$ *either* $\mathrm{SO}^0(p,q)$ *or one of the following groups with its standard representation:*

dimension	*signatur*	*holonomy group*
$2m \geq 4$	$(2r,2s)$	$\mathrm{U}(r,s)$ *und* $\mathrm{SU}(r,s)$
$2m \geq 4$	(r,r)	$\mathrm{SO}(r,\mathbb{C})$
$4m \geq 8$	$(4r,4s)$	$\mathrm{Sp}(r,s)$ *und* $\mathrm{Sp}(r,s) \cdot \mathrm{Sp}(1)$
$4m \geq 8$	$(2r,2r)$	$\mathrm{Sp}(r,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$
$4m \geq 16$	$(4r,4r)$	$\mathrm{Sp}(r,\mathbb{C}) \cdot \mathrm{SL}(2,\mathbb{C})$
7	$(4,3)$	$\mathrm{G}^*_{2(2)}$
14	$(7,7)$	$\mathrm{G}^{\mathbb{C}}_2$
8	$(4,4)$	$\mathrm{Spin}(4,3)$
16	$(8,8)$	$\mathrm{Spin}(7,\mathbb{C})$

As one easily sees, this list does not contain a group in Lorentzian signature. This reflects a special algebraic fact concerning irreducibly acting connected subgroups of the Lorentzian group $\mathrm{O}(1,n-1)$ (cf. [DO01]).

Theorem 2.8 *If* $H \subset \mathrm{O}(1,n-1)$ *is a connected Lie subgroup acting irreducibly on* $\mathbb{R}^{1,n-1}$*, then* $H = \mathrm{SO}^0(1,n-1)$*.*

The proofs of the basic Theorems stated in this section can be found in [Sa89], [Jo00], [B09]. We refer to [Sa89] and [Jo00] also for constructions of Riemannian manifolds with special holonomy.

3 Lorentzian holonomy groups - the algebraic classification

In this section we will describe the algebraic classification of the reduced holonomy groups of Lorentzian manifolds.

In dimension 4 there are 14 types of Lorentzian holonomy groups which were discovered by Schell [Sch60] and Shaw [Sh70]. We will not recall this list here, besides to the original papers we refer to [H93], [HL00], [Be87] chap. 10, [GL08]. In the following we will consider arbitrary dimension.

Due to Theorem 2.4 and Theorem 2.8 the Decomposition Theorem for Lorentzian manifolds can be formulated as follows:

Theorem 3.1 *Let (N,h) be a simply connected, geodesically complete Lorentzian manifold. Then (N,h) is isometric to the product*

$$(N,h) \simeq (M,g) \times (M_1,g_1) \times \ldots \times (M_r,g_r),$$

where (M_i,g_i) are either flat or irreducible Riemannian manifolds and (M,g) is either

1. *$(\mathbb{R},-dt^2)$,*
2. *an irreducible Lorentzian manifold with holonomy group $\mathrm{SO}^0(1,n-1)$ or*
3. *a Lorentzian manifold with weakly irreducible holonomy representation which admits a degenerate invariant subspace.*

Since the holonomy groups of the Riemannian factors are known, it remains to classify the weakly irreducible Lorentzian holonomy representations which admit an invariant degenerate subspace.
Let (M,g) be a weakly irreducible, but non-irreducible Lorentzian manifold, and let $x \in M$. Then the holonomy representation $\rho : \mathrm{Hol}_x(M,g) \to \mathrm{O}(T_xM,g_x)$ admits a degenerate invariant subspace $W \subset T_xM$. The intersection $V := W \cap W^\perp \subset T_xM$ is a holonomy invariant light-like line. Hence the holonomy group $\mathrm{Hol}_x(M,g)$ lies in the stabilizer $\mathrm{O}(T_xM,g_x)_V$ of V in $\mathrm{O}(T_xM,g_x)$. Let us first describe this stabilizer more in detail[3].
We fix a basis $(f_1,,\ldots,f_n)$ in T_xM such that $f_1 \in V$ and

$$\Big(g_x(f_i,f_j)\Big) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

identify (T_xM,g_x) with the Minkowski space and and write the elements of $\mathrm{O}(T_xM,g_x)$ as matrices with respect to this basis. The stabilizer of the isotropic line $\mathbb{R}f_1 \subset \mathbb{R}^{1,n-1}$ is a semidirect product and given by the matrices

$$\begin{aligned} \mathrm{O}(1,n-1)_{\mathbb{R}f_1} &= \big(\mathbb{R}^* \times \mathrm{O}(n-2)\big) \ltimes \mathbb{R}^{n-2} \\ &= \left\{ \begin{pmatrix} a^{-1} & x^t & -\frac{1}{2}a\|x\|^2 \\ 0 & A & -aAx \\ 0 & 0 & a \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R}^* \\ x \in \mathbb{R}^{n-2} \\ A \in \mathrm{O}(n-2) \end{array} \right\}. \end{aligned}$$

The Lie algebra of $\mathrm{O}(1,n-1)_{\mathbb{R}f_1}$ is

$$\begin{aligned} \mathfrak{so}(1,n-1)_{\mathbb{R}f_1} &= \big(\mathbb{R} \oplus \mathfrak{so}(n-2)\big) \ltimes \mathbb{R}^{n-2} \\ &= \left\{ \begin{pmatrix} \alpha & y^t & 0 \\ 0 & X & -y \\ 0 & 0 & -\alpha \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha \in \mathbb{R} \\ y \in \mathbb{R}^{n-2} \\ X \in \mathfrak{so}(n-2) \end{array} \right\}. \end{aligned}$$

[3]The connected component of the stabilizer $\mathrm{O}(1,n-1)_{\mathbb{R}f_1} \subset \mathrm{O}(1,n-1)$ of a light-like line $\mathbb{R}f_1$ is isomorphic to the group of similarity transformation of the Euclidian space $\mathbb{R}^{n-2}$, i.e., to the group generated by translations, dilatations and rotations of $\mathbb{R}^{n-2}$. Therefore, in some papers this group is denoted by $\mathrm{Sim}(n-2)$.

Let us denote a matrix in the Lie algebra $\mathfrak{so}(1,n-1)_{\mathbb{R}f_1}$ by (α, X, y) (in the obvious way). The commutator is given by

$$[(\alpha,X,y),(\beta,Y,z)] = \big(0,[X,Y],(X+\alpha\,\mathrm{Id})z-(Y+\beta\,\mathrm{Id})y\big),$$

which describes the semi-direct structure. In particular, $\mathbb{R}$, $\mathbb{R}^{n-2}$ and $\mathfrak{so}(n-2)$ are subalgebras of $\mathfrak{so}(1,n-1)_{\mathbb{R}f_1}$. Now, one can assign to any subalgebra $\mathfrak{h}\subset\mathfrak{so}(1,n-1)_{\mathbb{R}f_1}$ the projections $\mathrm{pr}_{\mathbb{R}}(\mathfrak{h})$, $\mathrm{pr}_{\mathbb{R}^{n-2}}(\mathfrak{h})$ and $\mathrm{pr}_{\mathfrak{so}(n-2)}(\mathfrak{h})$ onto these parts. The subalgebra

$$\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n-2)}(\mathfrak{h}) \subset \mathfrak{so}(n-2)$$

is called the *orthogonal part of* $\mathfrak{h}$. $\mathfrak{g}$ is reductive, i.e. its Levi decomposition is given by $\mathfrak{g}=\mathfrak{z}(\mathfrak{g})\oplus[\mathfrak{g},\mathfrak{g}]$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and the commutator $[\mathfrak{g},\mathfrak{g}]$ is semi-simple.

The first step in the classification of weakly irreducible holonomy representations is a result due to L. Berard-Bergery and A. Ikemakhen (cf. [BI93]), who described the possible algebraic types of weakly irreducibly acting subalgebras $\mathfrak{h}$ of the stabilizer $\mathfrak{so}(1,n-1)_{\mathbb{R}f_1}=\big(\mathbb{R}\oplus\mathfrak{so}(n-2)\big)\ltimes\mathbb{R}^{n-2}$. A geometric proof of this result was later given by A. Galaev in [G06a].

Theorem 3.2 *Let $f_1\in\mathbb{R}^{1,n-1}$ be a light-like vector in the Minkowski space and let*

$$\mathfrak{h}\subset\mathfrak{so}(1,n-1)_{\mathbb{R}f_1}=(\mathbb{R}\oplus\mathfrak{so}(n-2))\ltimes\mathbb{R}^{n-2}$$

be a weakly-irreducible subalgebra of the stabilizer of $\mathbb{R}f_1$ in $\mathfrak{so}(1,n-1)$. We denote by $\mathfrak{g}:=\mathrm{pr}_{\mathfrak{so}(n-2)}(\mathfrak{h})\subset\mathfrak{so}(n-2)$ the orthogonal part of $\mathfrak{h}$. Then $\mathfrak{h}$ is of one of the following four types:

1. $\mathfrak{h}=\mathfrak{h}^1(\mathfrak{g}) := (\mathbb{R}\oplus\mathfrak{g})\ltimes\mathbb{R}^{n-2}$.
2. $\mathfrak{h}=\mathfrak{h}^2(\mathfrak{g}) := \mathfrak{g}\ltimes\mathbb{R}^{n-2}$.
3. $\mathfrak{h}=\mathfrak{h}^3(\mathfrak{g},\varphi) := \big\{\big(\varphi(X),X+Y,z\big)\mid X\in\mathfrak{z}(\mathfrak{g}),\,Y\in[\mathfrak{g},\mathfrak{g}],\,z\in\mathbb{R}^{n-2}\big\}$,
   where $\varphi:\mathfrak{z}(\mathfrak{g})\to\mathbb{R}$ is a linear and surjective map.
4. $\mathfrak{h}=\mathfrak{h}^4(\mathfrak{g},\psi) := \big\{\big(0,X+Y,\psi(X)+z\big)\mid X\in\mathfrak{z}(\mathfrak{g}),\,Y\in[\mathfrak{g},\mathfrak{g}],\,z\in\mathbb{R}^{k}\big\}$,
   where $\mathbb{R}^{n-2}=\mathbb{R}^m\oplus\mathbb{R}^k$, $0<m<n-2$,
   $\mathfrak{g}\subset\mathfrak{so}(\mathbb{R}^k)$,
   $\psi:\mathfrak{z}(\mathfrak{g})\to\mathbb{R}^m$ *linear and surjective.*

In the following we will refer to theses cases as the Lie algebras $\mathfrak{h}$ of type 1 - type 4. The types 1 and 2 are called *uncoupled types*, the types 3 and 4 *coupled types*, since the $\mathfrak{so}(n-2)$- part is coupled by φ and ψ with the $\mathbb{R}$- and the $\mathbb{R}^{n-2}$-part, respectively.

Theorem 3.2 reduces the classification of Lorentzian holonomy algebras to the description of the *orthogonal part* $\mathfrak{g}=\mathrm{pr}_{\mathfrak{so}(n-2)}\,\mathfrak{hol}(M,g)$. First, let us look at the geometric meaning of $\mathfrak{g}$. Let $\mathcal{V}\subset TM$ be the holonomy distribution, defined by $\mathbb{R}f_1\subset T_xM$. The orthogonal complement $\mathcal{V}^\perp\subset TM$ is parallel as well and contains $\mathcal{V}$. Hence, $(\mathcal{V}^\perp/\mathcal{V},\widetilde{g},\widetilde{\nabla}^g)$ is a vector bundle of rank $(n-2)$ on M equipped with a positive definite bundle metric $\widetilde{g}$, induced by g, and a metric covariant derivative $\widetilde{\nabla}^g$, induced by the Levi-Civita-connection of g. It is not difficult to check, that the holonomy algebra of $(\mathcal{V}^\perp/\mathcal{V},\widetilde{\nabla}^g)$ coincides with $\mathfrak{g}$:

Proposition 3.1 ([L03]) *Let* (M,g) *be a Lorentzian manifold with a parallel light-like distribution* $\mathscr{V} \subset TM$*. Then,*

$$\mathfrak{hol}_x(\mathscr{V}^\perp/\mathscr{V}, \widetilde{\nabla}^g) = \mathrm{pr}_{\mathfrak{so}(n-2)}\, \mathfrak{hol}_x(M,g).$$

Moreover, the different types of holonomy algebras translate into special curvature properties of the light-like hypersurface of M, defined by the involutive distribution $\mathscr{V}^\perp$. For details we refer to [Bez05].

Thomas Leistner studied the orthogonal part of $\mathfrak{hol}(M,g)$ and obtained the following deep result, cf. [L03], [L07].

Theorem 3.3 *Let* (M^n,g) *be a Lorentzian manifold with a weakly irreducible but non-irreducible holonomy group* $\mathrm{Hol}^0(M,g)$*. Then the orthogonal part* $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n-2)}(\mathfrak{hol}(M,g))$ *of the holonomy algebra is the holonomy algebra of a Riemannian manifold.*

Leistner's proof of this Theorem is based on the observation of a special algebraic property of the orthogonal part $\mathfrak{g}$ of a Lorentzian holonomy algebra. It is a so-called *weak Berger algebra* - a notion, which was introduced and studied by T. Leistner in [L02a] (see also [L03], [L07], [G05]). We will explain this notion here shortly.
Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a subalgebra of the linear maps of a finite dimensional real or complex vector space V with scalar product $\langle\cdot,\cdot\rangle$. Then we consider the following spaces

$$\begin{aligned} \mathscr{K}(\mathfrak{g}) &:= \{R \in \Lambda^2(V^*) \otimes \mathfrak{g} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}, \\ \mathscr{B}(\mathfrak{g}) &:= \{B \in V^* \otimes \mathfrak{g} \mid \langle B(x)y,z\rangle + \langle B(y)z,x\rangle + \langle B(z)x,y\rangle = 0\}. \end{aligned}$$

The space $\mathscr{K}(\mathfrak{g})$ is called *the space of curvature tensors*[4] *of* $\mathfrak{g}$. $\mathscr{B}(\mathfrak{g})$ is called *space of weak curvature tensors of* $\mathfrak{g}$. Now, let $\mathfrak{g}$ be an orthogonal Lie algebra, i.e., $\mathfrak{g} \subset \mathfrak{so}(V,\langle\cdot,\cdot\rangle)$. Then any curvature tensor R of $\mathfrak{g}$ satisfies in addition the symmetry properties

$$\begin{aligned} \langle R(x,y)u,v\rangle &= -\langle R(x,y)v,u\rangle, \\ \langle R(x,y)u,v\rangle &= +\langle R(u,v)x,y\rangle. \end{aligned}$$

Hence, for each $R \in \mathscr{K}(\mathfrak{g})$ and $x \in V$ we have $R(x,\cdot) \in \mathscr{B}(\mathfrak{g})$.
A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is called *Berger algebra* if there are enough curvature tensors to generate $\mathfrak{g}$, i.e., if

$$\mathfrak{g} = \mathrm{span}\{R(x,y) \mid x,y \in V, R \in \mathscr{K}(\mathfrak{g})\}.$$

An orthogonal Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V,\langle\cdot,\cdot\rangle)$ is called *weak Berger algebra* if there are enough weak curvature tensors to generate $\mathfrak{g}$, i.e., if

$$\mathfrak{g} = \mathrm{span}\{B(x) \mid x \in V, B \in \mathscr{B}(\mathfrak{g})\}.$$

Obviously, every orthogonal Berger algebra is a weak Berger algebra. For an *Euclidian* space V, the Bianchi identity defining $\mathscr{B}(\mathfrak{g})$ is used to prove the following decomposition property of the space of weak curvature tensors $\mathscr{B}(\mathfrak{g})$:

Proposition 3.2 *Let V be an Euclidian space and let* $\mathfrak{g} \subset \mathfrak{so}(V)$ *be a weak Berger algebra. Then V decomposes into orthogonal* $\mathfrak{g}$*-invariant subspaces*

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_s,$$

[4]This notation is motivated by the fact, that the condition which defines $\mathscr{K}(\mathfrak{g})$ is just the Bianchi identity for the curvature tensor R^∇_x of a torsion free covariant derivative ∇.

where $\mathfrak{g}$ acts trivial on V_0 (possibly 0-dimensional) and irreducible on V_j, $j = 1, \ldots, s$. Moreover, $\mathfrak{g}$ is the direct sum of ideals

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s, \tag{1}$$

where $\mathfrak{g}_j$ acts irreducible on V_j and trivial on V_i if $i \neq j$. $\mathfrak{g}_j \subset \mathfrak{so}(V_j)$ is a weak Berger algebra and $\mathscr{B}(\mathfrak{g}) = \mathscr{B}(\mathfrak{g}_1) \oplus \ldots \oplus \mathscr{B}(\mathfrak{g}_s)$.

Now, let (M,g) be a Lorentzian manifold with holonomy group $\mathrm{Hol}_x(M,g)$. From the Ambrose-Singer Theorem 2.1 it follows that the holonomy algebra $\mathfrak{hol}_x(M,g) \subset \mathfrak{so}(T_xM, g_x)$ is a Berger algebra. Moreover, looking more carefully at the curvature endomorphisms one obtains:

Proposition 3.3 *Let (M^n, g) be a Lorentzian manifold with a weakly irreducible but non-irreducible holonomy group $\mathrm{Hol}^0_x(M,g)$. Then the orthogonal part $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n-2)}(\mathfrak{hol}_x(M,g))$ of the holonomy algebra is a weak Berger algebra on an Euclidean space. Hence it decomposes into a direct sum of irreducibly acting weak Berger algebras.*

Using representation and structure theory of semi-simple Lie algebras, T. Leistner proved the following central Theorem which implies Theorem 3.3.

Theorem 3.4 *Any irreducible weak Berger algebra on an Euclidian space is the holonomy algebra of an irreducible Riemannian manifold.*

All together we obtain the following classification Theorem for Lorentzian holonomy groups.

Theorem 3.5 (The connected holonomy groups of Lorentzian manifolds) *Let (M,g) be an n-dimensional, simply connected, indecomposable Lorentzian manifold. Then either (M,g) is irreducible and the holonomy group is the Lorentzian group $\mathrm{SO}^0(1,n-1)$, or the holonomy group lies in the stabilizer $\mathrm{SO}^0(1,n-1)_V = (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \ltimes \mathbb{R}^{n-2}$ of a light-like line V. In the second case, let $G' \subset G \subset \mathrm{SO}(n-2)$ be the closed subgroups with Lie algebras $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n-2)} \mathfrak{hol}_x(M,g) \subset \mathfrak{so}(n-2)$, respectively. Then $G \subset \mathrm{SO}(n-2)$ is the holonomy group of a Riemannian manifold and $\mathrm{Hol}_x(M,g)$ is of one of the following types:*

1. $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^{n-2}$.
2. $G \ltimes \mathbb{R}^{n-2}$.
3. $L \cdot G' \ltimes \mathbb{R}^{n-2}$, *where $L \subset \mathbb{R} \times \mathrm{SO}(n-2)$ is the connected Lie group with Lie algebra $\mathfrak{l} := \{(\varphi(X), X, 0) \mid X \in \mathfrak{z}(\mathfrak{g})\}$ for a surjective linear map $\varphi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{R}$.*
4. $\hat{L} \cdot G' \ltimes \mathbb{R}^{n-2-m}$, *where $\hat{L} \subset \mathrm{SO}(n-2) \ltimes \mathbb{R}^m$ is the connected Lie-group with Lie algebra $\hat{\mathfrak{l}} := \{(0, X, \psi(X)) \mid X \in \mathfrak{z}(\mathfrak{g})\}$ for a surjective linear map $\psi : \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathbb{R}^m$.*

4 Local realization of Lorentzian holonomy groups

In this section we will show, that *any* of the groups in the list of Theorem 3.5 can be realized as holonomy group of a Lorentzian manifold.

Lorentzian metrics with holonomy group of uncoupled type 1 and 2 are rather easy to construct:

Proposition 4.1 *Let (F,h) be a connected $(n-2)$-dimensional Riemannian manifold, and let $H \in C^\infty(\mathbb{R}\times F\times\mathbb{R})$ be a smooth function such that the Hessian of $H(0,\cdot,0)\in C^\infty(F)$ is non-degenerate in $x\in F$. Then the holonomy group of the Lorentzian manifold (M,g)*

$$M := \mathbb{R}\times F\times\mathbb{R}\ ,\ \ g := 2dvdu + H\,du^2 + h, \tag{2}$$

where v,u denote the coordinates of the $\mathbb{R}$-factors, is given by

$$\mathrm{Hol}_{(0,x,0)}(M,g) = \begin{cases} \mathrm{Hol}_x(F,h)\ltimes\mathbb{R}^{n-2} & if \quad \frac{\partial H}{\partial v}=0, \\ (\mathbb{R}^+\times\mathrm{Hol}_x(F,h))\ltimes\mathbb{R}^{n-2} & if \quad \frac{\partial^2 H}{\partial v^2}\neq 0. \end{cases}$$

This Theorem can be proved by direct calculation of the group of parallel displacements (cf. for example [B09], chap. 5). It is more difficult to produce metrics with the holonomy groups of coupled types 3 and 4.

The basic observation for constructing local metrics is the existence of adapted coordinates for Lorentzian manifolds with special holonomy, called *Walker coordinates*. Let (M,g) be a Lorentzian manifold with holonomy group acting weakly irreducible, but non-irreducible. Then, as we know from the previous section, there exists a 1-dimensional parallel light-like distribution $\mathscr{V}\subset TM$. Locally, the distribution $\mathscr{V}$ is spanned by a recurrent light-like vector field ξ, where a vector field ξ on (M,g) is called *recurrent* if there is a 1-form ω such that

$$\nabla^g\xi = \omega\otimes\xi.$$

A.G. Walker ([Wal50]) proved the existence of adapted coordinates in the presence of a parallel light-like line.

Proposition 4.2 (Walker coordinates) *Let (M,g) be an n-dimensional Lorentzian manifold with a parallel light-like line $\mathscr{V}\subset TM$. Then around any point $p\in M$ there are coordinates $(U,(v,x_1,\dots,x_{n-2},u))$ such that $g_{|U}$ has the form*

$$g_{|U} = 2dvdu + 2\sum_{i=1}^{n-2}A_i dx_i du + H\,du^2 + \sum_{j,k=1}^{n-2}h_{jk}dx_j dx_k, \tag{3}$$

where A_i, h_{jk} are smooth functions of $x_1,\dots,x_{n-2},u$ and the function H depends smoothly on the coordinates $v,x_1,\dots,x_{n-2},u$.

In these coordinates $\frac{\partial}{\partial v}$ generates the distribution $\mathscr{V}$ and $\frac{\partial}{\partial v},\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-2}}$ generate $\mathscr{V}^\perp$. The vector field $\frac{\partial}{\partial v}$ is parallel if H does not depend on v. We call a metric of the form (3) a *Walker metric*. The metrics (2) are special cases of Walker metrics, the holonomy group in this example is produced by the function H and the Riemannian metric h. In the following construction due to A. Galaev (cf. [G06b], [GL08]), the functions H and A_i in the Walker metric (3) are used to produce the holonomy groups in Theorem 3.5.

We consider the situation as described in section 3. Let $\mathfrak{g}\subset\mathfrak{so}(n-2)$ be the holonomy algebra of a Riemannian manifold. We will describe a Walker metric g of the form (3) on $\mathbb{R}^n$, such that $\mathfrak{hol}_0(\mathbb{R}^n,g)$ is of the form $\mathfrak{h}^1(\mathfrak{g})$, $\mathfrak{h}^2(\mathfrak{g})$, $\mathfrak{h}^3(\mathfrak{g},\varphi)$ and $\mathfrak{h}^4(\mathfrak{g},\psi)$, respectively, as described in Theorem 3.2. As we know, $\mathbb{R}^{n-2}$ has a decomposition into orthogonal subspaces

$$\mathbb{R}^{n-2} = \mathbb{R}^{n_0}\times\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_s}, \tag{4}$$

where $\mathfrak{g}$ acts trivial on $\mathbb{R}^{n_0}$ and irreducible on $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_s}$ and $\mathfrak{g}$ spits into a direct sum of ideals

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s,$$

where $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$ are holonomy algebras of irreducible Riemannian manifolds. Now, let $(e_1,\dots,e_{n-2})$ be an orthonormal basis of $\mathbb{R}^{n-2}$ adapted to the decomposition (4). We choose weak curvature endomorphisms $Q_I \in \mathscr{B}(\mathfrak{g})$, $I = 1,\dots,N$, which generate $\mathscr{B}(\mathfrak{g})$. Note that $Q_I(e_i) = 0$ for $i = 1,\dots,n_0$.

Let $\varphi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{R}$ and $\psi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{R}^m \subset \mathbb{R}^{n_0}$ be surjective linear maps. We extend φ and ψ to $\mathfrak{g}$ by setting $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$, $\psi|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and define the numbers:

$$\varphi_{Ii} \ := \ \frac{1}{(I-1)!}\varphi(Q_I(e_i)) \tag{5}$$

$$\psi_{Iij} \ := \ \frac{1}{(I-1)!}\Big\langle \psi(Q_I(e_i)), e_j \Big\rangle_{\mathbb{R}^{n-2}}, \tag{6}$$

where $I = 1,\dots,N$, $i = n_0+1,\dots,n-2$, $j = 1,\dots,m$. Then, one can realize any connected Lorentzian holonomy group by a Walker metric with polynomals as coefficients in the metric (3).

Theorem 4.1 ([G06b]) *Let $\mathfrak{h} \subset \mathfrak{so}(1,n-1)$ be one of the Lie algebras $\mathfrak{h}^1(\mathfrak{g})$, $\mathfrak{h}^2(\mathfrak{g})$, $\mathfrak{h}^3(\mathfrak{g},\varphi)$, $\mathfrak{h}^4(\mathfrak{g},\psi)$ in the list of Theorem 3.2, where $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n-2)}\mathfrak{h} \subset \mathfrak{so}(n-2)$ is the holonomy algebra of a Riemannian manifold. We consider the following Walker metric g on $\mathbb{R}^n$:*

$$g = 2dvdu + 2\sum_{i=1}^{n-2} A_i dx_i du + H\, du^2 + \sum_{i=1}^{n-2} dx_i^2,$$

where the functions A_i are given by

$$A_i(x_1,\dots,x_{n-2},u) \ := \ \sum_{I=1}^{N}\sum_{k,l=1}^{n-2} \frac{1}{3(I-1)!}\Big\langle Q_I(e_k)e_l + Q_I(e_l)e_k, e_i \Big\rangle_{\mathbb{R}^{n-2}} x_k x_l u^I,$$

and the function $H(v,x_1,\dots,x_{n-2},u)$ is defined in the following list, corresponding to the type of $\mathfrak{h}$:

$\mathfrak{h}$	H
Type 1: $\mathfrak{h}^1(\mathfrak{g}) = (\mathbb{R}\oplus\mathfrak{g}) \ltimes \mathbb{R}^{n-2}$	$v^2 + \sum_{i=1}^{n_0} x_i^2$
Type 2: $\mathfrak{h}^2(\mathfrak{g}) = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$	$\sum_{i=1}^{n_0} x_i^2$
Type 3: $\mathfrak{h}^3(\mathfrak{g},\varphi)$	$2v\sum_{I=1}^{N}\sum_{i=n_0+1}^{n-2} \varphi_{Ii} x_i u^{I-1} + \sum_{k=1}^{n_0} x_k^2$
Type 4: $\mathfrak{h}^4(\mathfrak{g},\psi)$	$2\sum_{I=1}^{N}\sum_{i=n_0+1}^{n-2}\sum_{j=1}^{m} \psi_{Iib} x_i x_j u^{I-1} + \sum_{k=m+1}^{n_0} x_k^2$

.

Then, $\mathfrak{h}$ is the holonomy algebra of $(\mathbb{R}^n, g)$ with respect to the point $0 \in \mathbb{R}^n$.

The proof of this Theorem uses that g is analytic. In this case, the holonomy algebra $\mathfrak{hol}_0(\mathbb{R}^n, g)$ is generated by the curvature tensor and its derivatives in the point $0 \in \mathbb{R}^n$. One calculates for the derivatives of the curvature tensor R:

$$\begin{aligned}
\mathrm{pr}_{\mathfrak{so}(n-2)}\left[\left(\nabla^{I-1}_{\partial_u} R\right)_0 (\partial_{x_i}, \partial_u)\right] &= Q_I(e_i), \\
\mathrm{pr}_{\mathbb{R}}\left[R_0 (\partial_v, \partial_u)\right] &= \frac{1}{2} \frac{\partial^2 H}{(\partial v)^2}(0), \\
\mathrm{pr}_{\mathbb{R}}\left[\left(\nabla^{I-1}_{\partial_u} R\right)_0 (\partial_{x_i}, \partial_u)\right] &= \frac{1}{2} \frac{\partial^{I+1} H}{\partial v \partial x_i (\partial u)^{I-1}}(0), \\
\mathrm{pr}_{\mathbb{R}^{n-2}}\left[\left(\nabla^{I-1}_{\partial_u} R\right)_0 (\partial_{x_a}, \partial_u)\right] &= \frac{1}{2} \sum_{j=1}^{n_0} \frac{\partial^{I+1} H}{\partial x_a \partial x_j (\partial u)^{I-1}}(0) \cdot e_j.
\end{aligned}$$

Hereby $I = 1, \ldots, N$, $i = n_0 + 1, \ldots, n$ and $a = 1, \ldots, n_0$. The first formula describes the only non-vanishing orthogonal parts of the curvature tensor and its derivatives in $0 \in \mathbb{R}^n$. Since $\mathfrak{g}$ is a weak Berger algebra and $\{Q_I \mid I = 1, \ldots, N\}$ generate $\mathscr{B}(\mathfrak{g})$, the orthogonal part of $\mathfrak{hol}_0(\mathbb{R}^n, g)$ coincide with $\mathfrak{g}$. The inclusion $\mathbb{R}^{n_0} \subset \mathfrak{hol}_0(\mathbb{R}^n, g)$ follows from the last formula, the proof that $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_s} \subset \mathfrak{hol}_0(\mathbb{R}^n, g)$ uses the fact, that $\mathfrak{g}_i$ acts irreducible on $\mathbb{R}^{n_i}$. The $\mathbb{R}$-part in $\mathfrak{hol}_0(\mathbb{R}^n, g)$ is generated by H, as the second and third formula show.

Recently, Ya. V. Bazaikin ([Ba09]) constructed coupled holonomy groups of type 3 and 4 using Walker metrics on $M := \mathbb{R} \times F \times \mathbb{R}$ of the form

$$g := \quad 2dvdu + Hdu^2 + 2A \odot du + h, \tag{7}$$

where (F, h) is a Riemannian manifold, A is an 1-form on F and H is a function on M. Moreover, he discussed causality properties, in particular global hyperbolicity of such metrics (see also section 5).

Finally, let us discuss Lorentzian manifolds (M, g) with the abelian and solvable reduced holonomy group

$$\mathrm{Hol}^0(M, g) = \begin{cases} \mathbb{R}^{n-2} \\ \mathbb{R}^+ \ltimes \mathbb{R}^{n-2}. \end{cases}$$

A Lorentzian manifold is called a *pp-wave*, if it admits a light-like parallel vector field ξ and if its curvature tensor R satisfies

$$R(X_1, X_2) = 0 \quad \text{for all } X_1, X_2 \in \xi^{\perp}. \tag{8}$$

A Lorentzian manifold is called a *pr-wave*, if it admits a light-like recurrent vector field ξ and if the curvature tensor satisfies (8). There are several equivalent conditions to (8), for which we refer to [L06a]. A pp-wave resp. a pr-wave is locally isometric to $(\mathbb{R}^n, g_H)$, where g_H is the Walker metric

$$g_H = 2dvdu + Hdu^2 + \sum_{i=1}^{n-2} dx_i^2$$

and, in case of a pp-wave, the function $H = H(v, x_1, \ldots, x_{n-2}, u)$ does not depend on v.

Proposition 4.3 ([L06a][L06b]) *Let (M, g) be a Lorentzian manifold with a light-like parallel (resp. recurrent) vector field. Then (M, g) is a pp-wave (resp. pr-wave) if and only if its holonomy group $\mathrm{Hol}^0(M, g)$ is contained in the abelian subgroup $\mathbb{R}^{n-2} \subset \mathrm{SO}(1, n-1)$ (resp. the solvable subgroup $\mathbb{R}^+ \ltimes \mathbb{R}^{n-2} \subset \mathrm{SO}(1, n-1)$).*

5 Global models with special Lorentzian holonomy

In the previous section we showed that any Lorentzian holonomy group can be realized by a local metric which is polynomial in the coordinates. In this section we will discuss some global constructions.

5.1 Lorentzian symmetric spaces

In the Riemannian case, the holonomy list is divided into the symmetric and the non-symmetric case. In Lorentzian signature, this distinction plays no essential role, since there are only few isometry classes of simply connected indecomposable Lorentzian symmetric spaces. We will shortly discuss these spaces and their holonomy groups.

Let (M^n,g) be a Lorentzian symmetric space. One has the following structure result:

Theorem 5.1 ([CW70]) *Let (M^n,g) be an indecomposable Lorentzian symmetric space of dimension $n \geq 2$. Then the transvection group of (M^n,g) is either semi-simple or solvable.*

We call a symmetric space *solvable* or *semi-simple* if its transvection group has this property. First we describe the *solvable* Lorentzian symmetric spaces.

Let $\underline{\lambda} = (\lambda_1,\dots,\lambda_{n-2})$ be an $(n-2)$-tupel of real numbers $\lambda_j \in \mathbb{R}\backslash\{0\}$ and let us denote by $M^n_{\underline{\lambda}}$ the Lorentzian space $M^n_{\underline{\lambda}} := (\mathbb{R}^n, g_{\underline{\lambda}})$, where $g_{\underline{\lambda}}$ is the Walker metric

$$g_{\underline{\lambda}} := 2dv\,du + \sum_{i=1}^{n-2} \lambda_i x_i^2\, du^2 + \sum_{i=1}^{n-2} dx_i^2. \tag{9}$$

If $\underline{\lambda}_\pi = (\lambda_{\pi(1)},\dots,\lambda_{\pi(n-2)})$ is a permutation of $\underline{\lambda}$ and $c>0$, then $M^n_{\underline{\lambda}}$ is isometric to $M^n_{c\underline{\lambda}_\pi}$. A direct calculation shows, that the space $M_{\underline{\lambda}}$ is geodesically complete and its curvature tensor is parallel. Hence the pp-wave $M_{\underline{\lambda}}$ is a Lorentzian symmetric space. These symmetric spaces were first described by M. Cahen and N. Wallach and are called now *Cahen-Wallach-spaces*. From Proposition 4.1 follows that the holonomy group of $M_{\underline{\lambda}}$ is the abelian subgroup $\mathbb{R}^{n-2} \subset (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \ltimes \mathbb{R}^{n-2}$. The transvection group of $M_{\underline{\lambda}}$ is solvable. For a description of this group we refer to [Ne03].

Theorem 5.2 ([CW70], [CP80]) *Let (M^n,g) be an indecomposable solvable Lorentzian symmetric space of dimension $n \geq 3$. Then (M^n,g) is isometric to $M^n_{\underline{\lambda}}/\Gamma$, where $\underline{\lambda} \in (\mathbb{R}\backslash\{0\})^{n-2}$ and Γ is a discrete subgroup of the centralizer $Z_{\underline{\lambda}}$ of the transvection group of $M_{\underline{\lambda}}$ in its isometry group.*

For the centralizer $Z_{\underline{\lambda}}$ M. Cahen and Y. Kerbrat proved:

Theorem 5.3 ([CK78]) *Let $\underline{\lambda} = (\lambda_1,\dots,\lambda_{n-2})$ be a tupel of non-zero real numbers.*

1. *If there is a positive λ_i or if there are two numbers λ_i, λ_j such that $\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}^2$, then $Z_{\underline{\lambda}} \simeq \mathbb{R}$ and $\varphi \in Z_{\underline{\lambda}}$ if and only if $\varphi(v,x,u) = (v+\alpha, x, u)$, $\alpha \in \mathbb{R}$.*
2. *Let $\lambda_i = -k_i^2 < 0$ and $\frac{k_i}{k_j} \in \mathbb{Q}$ for all $i,j \in \{1,\dots,n-2\}$. Then $\varphi \in Z_{\underline{\lambda}}$ if and only if*
$$\varphi(v,x,u) = (v+\alpha, (-1)^{m_1}x_1,\dots,(-1)^{m_{n-2}}x_{n-2}, u+\beta),$$
where $\alpha \in \mathbb{R}$, $m_1,\dots,m_{n-2} \in \mathbb{Z}$ and $\beta = \frac{m_i \cdot \pi}{k_i}$ for all $i = 1,\dots,n-2$.

Next, let us describe the *semi-simple* Lorentzian symmetric spaces. We denote by $S^n_1(r)$ the pseudo-sphere

$$S^n_1(r) := \{x \in \mathbb{R}^{1,n} \mid \langle x,x\rangle_{1,n} = -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2\}$$

and by $H^n_1(r)$ the pseudo-hyperbolic space

$$H^n_1(r) := \{x \in \mathbb{R}^{2,n-1} \mid \langle x,x\rangle_{2,n-1} = -x_1^2 - x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = -r^2\}$$

with the Lorentzian metrics induced by $\langle\cdot,\cdot\rangle_{1,n}$ and $\langle\cdot,\cdot\rangle_{2,n-1}$, respectively. $S^n_1(r)$ and $H^n_1(r)$ are semi-simple symmetric spaces of constant sectional curvature with full holonomy group $\mathrm{SO}^0(1,n-1)$. Moreover,

Theorem 5.4 ([CL90], [Wo84]) *Let (M^n,g) be an indecomposable semi-simple Lorentzian symmetric space of dimension $n \geq 3$. Then (M^n,g) has constant sectional curvature $k \neq 0$. Therefore, it is isometric to $S^n_1(r)/_{\{\pm I\}}$ or $S^n_1(r)$ if $k = \frac{1}{r^2} > 0$, or to a Lorentzian covering of $H^n_1(r)/_{\{\pm I\}}$ if $k = -\frac{1}{r^2} < 0$.*

5.2 Holonomy of Lorentzian cones

Cone constructions are often used to reduce a geometric problem on a manifold to a holonomy problem of the cone over that manifold. For example, C. Bär ([Ba93]) used this method to describe all Riemannian geometries with real Killing spinors. Other applications can be found in [BJ10], chapter 2, and in [M09b]. It is a classical result of S. Gallot ([Ga79]) that the holonomy group of the cone over a complete Riemannian manifold (N,h) is either irreducible or (N,h) has constant sectional curvature (which implies that the cone is flat). In the pseudo-Riemannian situation this is not longer true. The pseudo-Riemannian case was recently studied in [AC09]. We will describe the results of this paper for the Lorentzian cases here. There are two types of Lorentzian cones, the time-like cone $C_-(N,h)$ over a Riemannian manifold (N,h) and the space-like cone $C_+(N,h)$ over a Lorentzian manifold (N,h):

$$C_\varepsilon(N,h) := (\mathbb{R}^+ \times N\,,\, g_\varepsilon = \varepsilon\, dt^2 + t^2 h), \quad \varepsilon = \pm 1.$$

First, let us illustrate the difference to the Riemannian case with two examples.

1. Let (F,r) be a complete Riemannian manifold of dimension at least 2 which is not of constant sectional curvature. Then the Lorentzian manifold

   $$N := \mathbb{R} \times F\,,\; h := -ds^2 + \cosh^2(s) r$$

   is complete and not of constant sectional curvature, and the holonomy representation of its space-like cone $C_+(N,h)$ decomposes into proper non-degenerate invariant subspaces.

2. Let $(F,r) = (F_1,r_1) \times (F_2,r_2)$ be a product of a flat and a non-flat complete Riemannian manifold. Then the Riemannian manifold

   $$N := \mathbb{R} \times F\,,\; h := ds^2 + e^{-2s} r$$

   is complete and its time-like cone $C_-(N,h)$ is non-flat and has a light-like parallel vector field as well as a non-degenerate proper holonomy invariant subspace.

The structure of geodesically complete simply connected manifolds with non-irreducible Lorentzian cone is described in the following Theorem.

Theorem 5.5 ([AC09]) *Let (N,h) be a geodesically complete, simply connected Riemannian or Lorentzian manifold of dimension at least 2 with the corresponding Lorentzian cone $C_\varepsilon(N,h)$, $\varepsilon = \pm 1$.*

1. If the holonomy representation of the cone $C_\varepsilon(N,h)$ is decomposable, then

$$Hol_x(N,h) = \mathrm{SO}^0(T_xN, h_x).$$

If (N,h) is Riemannian (i.e. $\varepsilon = -1$), then (N,h) has either constant sectional curvature ε or is isometric to the product

$$(\mathbb{R}^+ \times N_1 \times N_2\,, -\varepsilon ds^2 + \cosh^2(s)h_1 + \sinh^2(s)h_2),$$

where (N_i,h_i) are Riemannian manifolds and (N_2,h_2) has constant curvature $-\varepsilon$ or dimension ≤ 1. If (N,h) is a Lorentzian manifold (i.e. $\varepsilon = 1$), then the same result is true on each connected component of a certain open dense set of N.

2. If the holonomy representation of $C_\varepsilon(N,h)$ is indecomposable but non-irreducible, then the cone admits a parallel light-like vector field.
If (N,h) is a Riemannian manifold, then (N,h) is isometric to

$$(\mathbb{R} \times F, -\varepsilon ds^2 + e^{-2s} r),$$

where (F,r) is a complete Riemannian manifold, and the holonomy group of the cone is given by

$$Hol(C_-(N,h)) = Hol(F,r) \ltimes \mathbb{R}^{\dim F}.$$

If (N,h) is a Lorentzian manifold, the same result is true for any connected component of a certain open sense set of N.

Note, that a compact pseudo-Riemannian manifold need not to be geodesically complete. A stronger result hold for *compact* manifolds (N,h).

Theorem 5.6 ([M09b]) *Let $C_\varepsilon(N,h)$ be the Lorentzian cone over a compact connected Riemannian or Lorentzian manifold (N,h). Then the holonomy representation of $C_\varepsilon(N,h)$ is indecomposable.*

In [AC09] and [M09a] Theorem 5.6 is proved under the additional assumption, that (N,g) is geodesically complete. It was first shown, that decomposability implies that (N,h) has constant sectional curvature ε. Since there are no compact de Sitter spaces, (N,h) has to be Riemannian with flat, but non-simply connected cone. In [M09b], Proposition 4.1., V. Matveev and P. Mounoud gave a nice short argument using only the compactness of (N,h) to show that the metric of a decomposable cone is definite.

5.3 Lorentzian metrics with special holonomy on non-trivial torus bundles

In this section we describe a construction of Lorentzian metrics with special holonomy on non-trivial torus bundles which is due to K. Lärz ([La10a], [La11]). The basic idea is to consider Lorentzian metrics on S^1-bundles which look like a Walker metric (see section

4). For that, let (N,h_N) be a Riemannian manifold, $\omega \in H^2(N,\mathbb{Z})$ and $\pi : M \to N$ the S^1-bundle with $c_1(M) = \omega$. For any closed 2-form ψ on N representing ω in the de Rham cohomology, there is a connection form $A : TM \to i\mathbb{R}$ on M with curvature $dA = -2\pi i \pi^* \psi$ (see e.g. [B09]). For a smooth function $f \in C^\infty(M)$ and a nowhere vanishing closed 1-form η on N we consider the following Lorentzian metric on M

$$g := 2iA \odot \pi^*\eta + f \cdot (\pi^*\eta)^2 + \pi^* h_N. \tag{10}$$

The vertical fundamental vector field ξ of the S^1-action on M is light-like. Using that η is closed, one obtains for the covariant derivative of ξ

$$\nabla^g_Z \xi = -\xi(f) \cdot \eta\big(d\pi(Z)\big) \cdot \xi\,, \qquad Z \in \mathfrak{X}(M).$$

This shows that the vertical tangent bundle $\mathscr{V} := \mathbb{R}\xi \subset TM$ is a parallel distribution, and that ξ is parallel iff f is constant on the fibres of π. Moreover, if $\xi(f) \neq 0$, the distribution $\mathscr{V}$ does not contain a parallel vector field. Hence, the holonomy representation of (M,g) has an invariant light-like vector resp. line.

First, let us mention that there are special cases of this construction where M is totally twisted, i.e., where M is not homeomorphic to $Y \times \mathbb{R}$ or $Y \times S^1$. Thereby, M can be compact as well as non-compact (cf. [La10a]).

Here we will consider a special case of this construction, where a 1-dimensional factor splits up: We take $N := B \times L$, with a 1-dimensional manifold L, $\eta := du$, where u is the coordinate of L, and $\omega \in H^2(B,\mathbb{Z})$. Then $M = \widetilde{M} \times L$, where $\widetilde{\pi} : \widetilde{M} \to B$ is the S^1-bundle on B with 1. Chern class ω. Now, let $\widetilde{A}$ be a connection form on $\widetilde{M}$, h a Riemannian metric on B and f a smooth function on M. Then the metric (10) has the special form

$$g := g_{f,\widetilde{A},h} := 2i\widetilde{A} \odot du + f \cdot du^2 + \widetilde{\pi}^* h.$$

We call $(M, g_{f,\widetilde{A},h})$ *a manifold of toric type over* (B,h)[5]. One can use this construction to produce Lorentzian manifolds with non-trivial topology and holonomy group

$$\mathrm{Hol}(M,g) = \begin{cases} G \ltimes \mathbb{R}^{n-2} \\ (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^{n-2} \end{cases},$$

where G is one of the groups $\mathrm{SO}(n-2)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$ or $\mathrm{Sp}(k)$. The horizontal lift $TB^* \subset TM$ of TB with respect to $\widetilde{A}$ is isomorphic to the vector bundle $\mathscr{V}^\perp/\mathscr{V}$. Looking at the parallel displacement along the horizonal lifts of curves in B, one can check that $\mathrm{Hol}(B,h) \subset \mathrm{Hol}(\mathscr{V}^\perp/\mathscr{V}, \widetilde{\nabla}^g)$. The projection $G := \mathrm{pr}_{\mathrm{O}(n-2)} \mathrm{Hol}(M,g) \subset \mathrm{O}(n-2)$ coincides with $\mathrm{Hol}(\mathscr{V}^\perp/\mathscr{V}, \widetilde{\nabla}^g)$. Hence, to ensure that the holonomy group $\mathrm{Hol}(\mathscr{V}^\perp/\mathscr{V}, \widetilde{\nabla}^g)$ is not larger then $\mathrm{Hol}(B,h)$, the bundle $\mathscr{V}^\perp/\mathscr{V}$ has to admit an additional $\widetilde{\nabla}^g$-parallel structure corresponding to the group G is question. This is possible for appropriate classes $\omega \in H^2(B,\mathbb{Z})$ defining the topological type of the S^1-bundle $\widetilde{M} \to B$ and appropriate closed 2-forms ψ representing ω and defining the connection form $\widetilde{A}$. We quote some of the results of K. Lärz.

Theorem 5.7 ([La10a], [La11]) *With the notations above and a sufficient generic function f in every case, we have:*

[5]Note, that if $L = S^1$, M is a torus bundle with one trivial direction

1. Let (B,h) be an $(n-2)$-dimensional Riemannian manifold such that $\mathrm{Hol}(B,h) = \mathrm{SO}(n-2)$. *If $(M, g_{f,\widetilde{A},h})$ is of toric type over (B,h), then*

$$\mathrm{Hol}(M, g_{f,\widetilde{A},h}) = \begin{cases} \mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2} & f \text{ fibre-constant on } \widetilde{P} \\ (\mathbb{R}^+ \times \mathrm{SO}(n-2)) \ltimes \mathbb{R}^{n-2} & \text{otherwise.} \end{cases}$$

2. Let (B^{2m},h,J) be a compact, simply connected, irreducible Kähler manifold with $c_1(B,J) < 0$ and let h be its Kähler-Einstein metric. Then, for any Hodge class $\omega \in H^{1,1}(B,\mathbb{Z}) := \mathrm{Im}\big(H^2(B,\mathbb{Z}) \to H^2(B,\mathbb{C})\big) \cap H^{1,1}(B,J)$,

$$\mathrm{Hol}(M, g_{f,\widetilde{A},h}) = \begin{cases} \mathrm{U}(m) \ltimes \mathbb{R}^{2m} & f \text{ fibre-constant on } \widetilde{P} \\ (\mathbb{R}^+ \times \mathrm{U}(m) \ltimes \mathbb{R}^{2m} & \text{otherwise.} \end{cases}$$

3. Let (B^{2m},J,h) be a Calabi-Yau manifold, i.e., a compact Kähler manifold with holonomy group $\mathrm{SU}(m)$. *Choose $\omega \in H^{1,1}(B,\mathbb{Z})$ and a harmonic representative $\psi \in \omega$ which in the case $L = S^1$ has integer values under the dual Lefschetz operator. Then*

$$\mathrm{Hol}(M, g_{f,\widetilde{A},h}) = \begin{cases} \mathrm{SU}(m) \ltimes \mathbb{R}^{2m} & f \text{ fibre-constant on } \widetilde{P} \\ (\mathbb{R}^+ \times \mathrm{SU}(m) \ltimes \mathbb{R}^{2m} & \text{otherwise.} \end{cases}$$

4. Let (B^{4k},J) be a holomorphic symplectic manifold with $b_2 \geq 4$ and Picard number $\rho(B,J) = b_2 - 2$. Then there exists an irreducible hyperkähler structure (B,J,J_2,J_3,h) with Kähler class in $H^2(B,\mathbb{Q})$ and $0 \neq \omega \in H^{1,1}(B,J) \cap H^{1,1}(B,J_2) \cap H^2(M,\mathbb{Z})$. Let $\psi \in \omega$ be a harmonic representative. Then

$$\mathrm{Hol}(M, g_{f,\widetilde{A},h}) = \begin{cases} \mathrm{Sp}(k) \ltimes \mathbb{R}^{4k} & f \text{ fibre-constant on } \widetilde{P} \\ (\mathbb{R}^+ \times \mathrm{Sp}(k)) \ltimes \mathbb{R}^{4k} & \text{otherwise.} \end{cases}$$

For a proof we refer to [La10a] and [La11]. There one can also find lots of concrete examples of the type described in the Theorem. In particular, this methods allows to construct spaces with disconnected holonomy groups.

Another bundle construction was considered by T. Krantz in [K10]. He studied S^1-bundles $\pi : M \to N$ over Riemannian manifolds (N,h) with Lorentzian Kaluza-Klein metrics on the total space of the form

$$g := A \odot A + \pi^* h,$$

where A is a connection form on M. In this case the fibre is time-like. Hereby, a parallel light-like distribution on (M,g) can occur only if the S^1-bundle admits a flat connection.

5.4 Geodesically complete and globally hyperbolic models

The bundle construction in section 5.3 produces compact as well as non-compact Lorentzian manifolds with special holonomy. Moreover, this bundle construction gives us complete compact examples: Let T^{n-1} be the flat torus with standard coordinates $(x_1,\ldots,x_{n-2},u)$ and take $\eta := du$ and $\psi := dx_1 \wedge du$. Consider the S^1-bundle $\pi : M \to T$ over T defined by $c_1(M) = [\psi]$, a connection form A on M with curvature $dA = -2\pi i \pi^* \psi$ and a smooth function f on T. Then the Lorentzian metric

$$g := 2iA \odot du + (f \circ \pi + 1)du^2 + \sum_{i=1}^{n-2} dx_i^2$$

on M is a geodesically complete and, if f is sufficient generic, (M,g) is indecomposable with abelian holonomy algebra $\mathbb{R}^{n-2}$ (cf. [La10a], Cor. 5.3.).

Examples of non-compact geodesically complete Lorentzian manifolds with special holonomy of type 2 can be found in papers of M. Sanchez, A. M. Candela and J. L. Flores (see [CFS03], [FS03]). These authors studied geodesics as well as causality properties for Lorentzian manifolds (M,g) of the form

$$M = \mathbb{R} \times F \times \mathbb{R}\,,\ g = 2dvdu + H(x,u)du^2 + h, \tag{11}$$

where (F,h) is a connected $(n-2)$-dimensional Riemannian manifold and H is a non-trivial smooth function. They call such manifolds *general plane-fronted waves (PFW)*. As we know from Proposition 4.1, if H is sufficient generic in a point, the holonomy group of the general plane-fronted wave is $\mathrm{Hol}(M,g) = \mathrm{Hol}(F,h) \ltimes \mathbb{R}^{n-2}$.

Proposition 5.1 ([CFS03]) *A general plane-fronted wave (11) is geodesically complete if and only if (F,h) is a complete Riemannian manifold and the maximal solutions $s \to x(s) \in F$ of the equation*

$$\frac{\nabla^F \dot{x}(s)}{ds} = \frac{1}{2}\,(\mathrm{grad}^F H)(x(s),s) \tag{12}$$

are defined on $\mathbb{R}$.

Equation (12) is studied in several cases. For example, if $H = H(x)$ is at most quadratic, i.e., if there is a point $x_0 \in F$ and constants $r > 0$ and $C > 0$ such that

$$H(x) \leq C\, d(x,x_0)^2 \qquad \text{for all } x \in F \text{ with } d(x,x_0) \geq r,$$

where $d(x,x_0)$ denotes the geodesic distance on (F,h), then the solutions of (12) are defined on $\mathbb{R}$. More on this subject can be found in [CFS03].

Another property, which is of special interest in Lorentzian geometry and analysis, is global hyperbolicity. A Lorentzian manifold is called *globally hyperbolic* if it is connected and time-oriented and admits a Cauchy surface, i.e., a subset S which is met by each inextendible time-like piecewise C^1-curve exactly once. For an introduction to this kind of Lorentzian manifolds, its relevance and equivalent definitions we refer to [BE96], [MS08], [BGP07], [Pf09]. In [BS05], A. Bernal and M. Sánchez proved a characterization of globally hyperbolic manifolds which is very useful for geometric purposes.

Proposition 5.2 ([BS05]) *A Lorentzian manifold is globally hyperbolic if and only if it is isometric to*

$$(\mathbb{R} \times S,\, g = -\beta dt^2 + g_t), \tag{13}$$

where β is a smooth positive function, g_t is a family of Riemannian metrics on S smoothly depending on $t \in \mathbb{R}$, and each $\{t\} \times S$ is a smooth space-like Cauchy hypersurface in M.

Under special conditions a general plane-fronted wave is globally hyperbolic.

Proposition 5.3 ([FS03]) *A general plane-fronted wave (11) is globally hyperbolic if (F,h) is complete and if the function $-H(x,u)$ is subquadratic at spacial infinity, i.e., if there is a point $x_0 \in F$ and continious functions $C_1(u) \geq 0$, $C_2(u) \geq 0$, $p(u) < 2$ such that*

$$-H(x,u) \leq C_1(u)\, d(x,x_0)^{p(u)} + C_2(u) \quad \textit{for all } (x,u) \in F \times \mathbb{R}.$$

In [Ba09] Y. V. Bazaikin constructed globally hyperbolic metrics of the more general form (7) and gave examples with holonomy of type 3 and 4. In section 6.1 we will discuss globally hyperbolic metrics with complete Cauchy surface and parallel spinors.

5.5 Topological properties

In [La10b] and [La11], Kordian Lärz studied topological properties of Lorentzian manifolds with special holonomy using Hodge theory of Riemannian foliations. We will briefly describe his results. Let (M,g) be a time-oriented Lorentzian manifolds with a 1-dimensional parallel light-like distribution $\mathscr{V} \subset TM$. Then there is a *global* recurrent vector field $\xi \in \Gamma(\mathscr{V})$. In [La10b], a Lorentzian manifold is called *decent*, if the vector field ξ can be chosen such that $\nabla_X \xi = 0$ for all $X \in \xi^\perp$. Now, fix a vector field Z on M satisfying

$$g(Z,Z) = 0 \text{ and } g(\xi,Z) = 1,$$

and denote by $S \subset TM$ the subbundle $S := \mathrm{span}(\xi,Z)^\perp$. Using the vector field Z we can define a Riemannian metric g^R on M by

$$g^R(\xi,\xi) := 1,\ g^R(Z,Z) := 1,\ g^R(\xi,Z) := 0,\ g^R_{|S\times S} := g_{|S\times S},\ \mathrm{span}(\xi,Z) \perp_{g^R} S.$$

Let $\mathscr{L}$ be the foliation of M in light-like curves given by the parallel line $\mathscr{V} \subset TM$ and let $\mathscr{L}^\perp$ be the foliation of M in light-like hypersurfaces given by the parallel subbundle $\mathscr{V}^\perp \subset TM$. If (M,g,ξ) is a decent spacetime, the Riemannian metric g^R is bundle-like with respect to the foliation $(M,\mathscr{L}^\perp)$. Moreover, if $L^\perp$ is a leaf of $\mathscr{L}^\perp$, then $g^R_{|TL^\perp\times TL^\perp}$ is bundle like with respect to $(L^\perp, \mathscr{L}_{|L^\perp})$ as well. Then, an application of Hodge theory and Weitzenböck formula for the twisted basic Hodge-Laplacian of Riemannian foliations yields the following result:

Proposition 5.4 ([La10b], [La11]) *Let (M,g) be a decent spacetime and suppose that the foliation $\mathscr{L}^\perp$ of M contains a compact leaf $L^\perp$ with $Ric(X,X) \geq 0$ for all $X \in TL^\perp$. Let $b_1(M)$ be the first Betti number of M.*

1. *If M is compact, then $1 \leq b_1(M) \leq \dim M$.*
2. *If M is non-compact and all leaves of $\mathscr{L}^\perp$ are compact, then $0 \leq b_1(M) \leq \dim M - 1$.*

Moreover, if $Ric_q(X,X) > 0$ for some $q \in L^\perp$ and all $X \in S_q$, the bounds are $1 \leq b_1(M) \leq 2$ and $0 \leq b_1(M) \leq 1$, respectively.

Explicit examples show, that the bounds for the 1. Betti number in Proposition 5.4 are sharp. If the foliation $\mathscr{L}^\perp$ of M admits a compact leaf with finite fundamental group, the holonomy algebra of (M,g) can only be of type 1, 2 or 3 (cf. Theorem 3.2), where the orthogonal part $\mathfrak{g}$ has an additional property. In special situations estimates for higher Betti numbers are possible (cf. [La10b], [La11]).

6 Lorentzian manifolds with special holonomy and additional structures

6.1 Parallel spinors

Now, let us consider a semi-Riemannian spin manifold (M,g) of signature (p,q) with spinor bundle S und spinor derivative ∇^S. We suppose in this review, that spin manifolds are space-and time-oriented. For a detailed introduction to pseudo-Riemannian spin geometry and the formulas for the spin representation see [B81] or [BK99]. In spin geometry one is interested in the description of all manifolds which admit parallel spinors, i.e., with spinor fields $\varphi \in \Gamma(S)$ such that $\nabla^S\varphi = 0$. This question is closely related to the holonomy group of (M,g), since the existence of parallel spinors restricts the holonomy group of (M,g). Let us explain this shortly. The spinor bundle is given by $S = Q\times_{(\mathrm{Spin}(p,q),\kappa)}\Delta_{p,q}$, where (Q,f) is a spin structure of (M,g) and $\kappa : \mathrm{Spin}(p,q) \to \mathrm{GL}(\Delta_{p,q})$ denotes the spinor representation in signature (p,q). Furthermore, let $\lambda : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ denote the double covering of the special orthogonal group by the spin group. We consider the holonomy group $\mathrm{Hol}_x(M,g)$ of (M,g) as a subgroup of $\mathrm{SO}(p,q)$ (by fixing a basis in T_xM). Using, that the spinor derivative is induced by the Levi-Civita connection, the holonomy principle gives:

Proposition 6.1 *1. If (M,g) admits a non-trivial parallel spinor, then there is an embedding $\iota : \mathrm{Hol}(M,g) \hookrightarrow \mathrm{Spin}(p,q)$ such that $\lambda\circ\iota = \mathrm{Id}_{\mathrm{Hol}(M,g)}$. Moreover, there exists a vector $v \in \Delta_{p,q}$ such that $\iota(\mathrm{Hol}(M,g)) \subset \mathrm{Spin}(p,q)_v$, where $\mathrm{Spin}(p,q)_v$ denotes the stabilizer of v under the action of the spin group. On the other hand, if there is an embedding $\iota : \mathrm{Hol}(M,g) \hookrightarrow \mathrm{Spin}(p,q)$ such that $\lambda\circ\iota = \mathrm{Id}_{\mathrm{Hol}(M,g)}$, then (M,g) admits a spin structure whose holonomy group is $\iota(\mathrm{Hol}(M,g))$. Moreover, if there is a spinor $v \in \Delta_{p,q}$ such that $\iota(\mathrm{Hol}(M,g)) \subset \mathrm{Spin}(p,q)_v$, than (M,g) admits a non-trivial parallel spinor field.*
2. If (M,g) is simply connected, then there is a bijective correspondence between the space of parallel spinors and the kernel of the action of the subalgebra $\lambda_*^{-1}(\mathfrak{hol}(M,g)) \subset \mathfrak{spin}(p,q)$ on $\Delta_{p,q}$:

$$\{\varphi\in\Gamma(S) \mid \nabla^S\varphi = 0\} \overset{1:1}{\Longleftrightarrow} \{v\in\Delta_{p,q} \mid \lambda_*^{-1}(\mathfrak{hol}(M,g))\,v = 0\}.$$

Using this Proposition, one can easily check which groups in the holonomy list allow the existence of parallel spinors. Let us first recall the results for Riemannian manifolds.

Theorem 6.1 *Let (M,g) be a Riemannian spin manifold of dimension $n\geq 2$ with non-trivial parallel spinor. Then (M,g) is Ricci-flat and non-locally symmetric. If (M,g) is irreducible and simply connected, the holonomy group is one of the groups $\mathrm{SU}(m)$ if $n = 2m\geq 4$, $\mathrm{Sp}(k)$ if $n = 4k \geq 8$, G_2 if $n = 7$, or $\mathrm{Spin}(7)$ if $n = 8$, with its standard representation.*

The list of holonomy groups in Theorem 6.1 was found by McK. Wang ([W89]). A list of the holonomy groups of irreducible, non-simply connected Riemannian spin manifolds with parallel spinors can be found in [SM00].

The situation in the Lorentzian case is a bit different. First of all, note that there are non-Ricci-flat as well as symmetric Lorentzian manifolds which admit parallel spinors. For example, let us consider the symmetric Cahen-Wallach spaces $M_{\underline{\lambda}} := (\mathbb{R}^n, g_{\underline{\lambda}})$ (cf. section 5.1, formula (9)). The Ricci-curvature of $M_{\underline{\lambda}}$ is given by

$$Ric(X) = -\sum_{j=1}^{n-2}\lambda_j\cdot g_{\underline{\lambda}}\Big(X,\frac{\partial}{\partial v}\Big)\cdot\frac{\partial}{\partial v}\,, \qquad X\in\mathfrak{X}(M_{\underline{\lambda}}).$$

If $\underline{\lambda} \neq (\lambda,\dots,\lambda)$, i.e., if $M_{\underline{\lambda}}$ is not locally conformally flat, then the space of parallel spinors on $M_{\underline{\lambda}}$ is $2^{[n/2]-1}$-dimensional (cf. [B00]).
Now, let us consider a Lorentzian spin manifold (M,g). Since (M,g) is time- and space-oriented, there is an indefinite hermitian bundle metric $\langle\cdot,\cdot\rangle$ on the spinor bundle S such

that

$$\begin{aligned} \langle X\cdot\varphi,\psi\rangle &= \langle\varphi, X\cdot\psi\rangle, \\ X(\langle\varphi,\psi\rangle) &= \langle\nabla^S_X\varphi,\psi\rangle + \langle\varphi,\nabla^S_X\psi\rangle \end{aligned}$$

for all vector fields X and spinor fields φ,ψ. If φ is a spinor field, the vector field V_φ defined by

$$g(X,V_\varphi) = -\langle X\cdot\varphi,\varphi\rangle$$

is future-directed and causal, i.e., $g(V_\varphi,V_\varphi)\le 0$. Moreover, V_φ has the same zeros as φ.

Proposition 6.2 *Let (M,g) be a Lorentzian spin manifold with a non-trivial parallel spinor field φ. Then the vector field V_φ is parallel and either time-like or light-like. Moreover, the Ricci-tensor of (M,g) satisfies*

$$Ric(X)\cdot\varphi = 0\,, \qquad X\in\mathfrak{X}(M).$$

Therefore, the Ricci-tensor is totally isotropic[6] *and the scalar curvature of (M,g) vanishes.*

Proposition 6.2 shows that the holonomy representation of a Lorentzian spin manifold with a parallel spinor acts trivial on a time-like or a light-like 1-dimensional subspace. Since a product of spin manifolds admits a parallel spinor if and only if its factors admit one, we obtain from the Decomposition Theorem of de Rham and Wu (Theorem 2.4):

Proposition 6.3 *Let (M,g) be a simply connected, geodesically complete Lorentzian spin manifold with non-trivial parallel spinor φ. Then (M,g) is isometric to the product*

$$\begin{array}{lll} & (\mathbb{R},-dt^2)\times(M_1,g_1)\times\ldots\times(M_k,g_k) & \text{if} \quad V_\varphi \text{ is time-like} \\ \text{or} & (N,h)\times(M_1,g_1)\times\ldots\times(M_k,g_k) & \text{if} \quad V_\varphi \text{ is light-like}, \end{array}$$

where (M_1,g_1), $\ldots$, (M_k,g_k) are flat or irreducible Riemannian spin manifolds with a parallel spinor and (N,h) is a weakly irreducible, but non-irreducible Lorentzian spin manifold with a parallel spinor.

Let us now consider a weakly irreducible Lorentzian spin manifolds (M,g) with parallel spinor. For small dimension, by studying the orbit structure of the spinor modul, R. Bryant [Bry00] and J. Figueroa-O'Farrill [F00] proved

Proposition 6.4 *The maximal stabilizer groups of a spinor $v\in\Delta_{1,n-1}$ with a light-like associated vector under the spin representation are*

$$\begin{array}{rcl} n\le 5 & : & 1\ltimes\mathbb{R}^{n-2} \\ n=6 & : & \mathrm{Sp}(1)\ltimes\mathbb{R}^4 \\ n=7 & : & (\mathrm{Sp}(1)\times 1)\ltimes\mathbb{R}^5 \\ n=8 & : & \mathrm{SU}(3)\ltimes\mathbb{R}^6 \\ n=9 & : & \mathrm{G}_2\ltimes\mathbb{R}^7 \\ n=10 & : & \mathrm{Spin}(7)\ltimes\mathbb{R}^8 \text{ and } \mathrm{SU}(4)\ltimes\mathbb{R}^8 \\ n=11 & : & (\mathrm{Spin}(7)\times 1)\ltimes\mathbb{R}^9 \text{ and } (\mathrm{SU}(4)\times 1)\ltimes\mathbb{R}^8. \end{array}$$

[6] This means $Ric(TM)\subset TM$ is a totally isotropic subspace.

Next, we explain the results of T. Leistner ([L02b], [L03]), who was able to determine all possible holonomy algebras of a weakly irreducible Lorentzian spin manifold (M,g) admitting a non-trivial parallel spinor using his holonomy classification. Since there is a parallel light-like vector field on (M,g), the holonomy algebra $\mathfrak{hol}(M,g)$ is of type 2 or 4, in particular, $\mathfrak{hol}(M,g) \subset \mathfrak{so}(n-2) \ltimes \mathbb{R}^{n-2}$. In order to determine $\mathfrak{hol}(M,g)$, one has to calculate the subalgebra $\lambda_*^{-1}(\mathfrak{hol}(M,g)) \subset \mathfrak{spin}(1,n-1)$ and the space

$$\{v \in \Delta_{1,n-1} \mid \lambda_*^{-1}(\mathfrak{hol}(M,g))\, v = 0\},$$

see Proposition 6.1. Let $\mathbb{R}^{1,n-1} = \mathbb{R} f_1 \oplus \mathbb{R}^{n-2} \oplus \mathbb{R} f_n$ be the decomposition of the Minkowski space as in section 3. We can identify the spinor moduls

$$\begin{array}{rcl} \Delta_{1,n-1} & = & \Delta_{n-2} \otimes \Delta_{1,1}, \\ v & = & v_1 \otimes u_1 + v_2 \otimes u_2, \end{array}$$

where (u_1, u_2) is a basis of $\Delta_{1,1} = \mathbb{C}^2$ and the Clifford multiplication with the isotropic vectors f_1 and f_n and with $x \in \mathbb{R}^{n-2}$ is given by

$$\begin{array}{rcl} f_1 \cdot (v_1 \otimes u_1 + v_2 \otimes u_2) & = & \sqrt{2} v_1 \otimes u_1 \\ f_n \cdot (v_1 \otimes u_1 + v_2 \otimes u_2) & = & -\sqrt{2} v_2 \otimes u_2 \\ x \cdot (v_1 \otimes u_1 + v_2 \otimes u_2) & = & (-x \cdot v_1) \otimes u_1 + (x \cdot v_2) \otimes u_2. \end{array}$$

For the covering map λ one calculates

$$\lambda_*^{-1}(\mathfrak{so}(n-2) \ltimes \mathbb{R}^{n-2}) = \mathfrak{spin}(n-2) + \{x \cdot f_1 \mid x \in \mathbb{R}^{n-2}\} \subset \mathfrak{spin}(1,n-1).$$

If $\mathfrak{h} \subset \mathfrak{so}(n-2) \ltimes \mathbb{R}^{n-2}$ acts weakly irreducible, there is a non-trivial vector $x \in \mathbb{R}^{n-2} \cap \mathfrak{h}$. Hence,

$$\{v \in \Delta_{1,n-1} \mid \lambda_*^{-1}(\mathfrak{h}) v = 0\} = \{v_2 \otimes u_2 \mid v_2 \in \Delta_{n-2} \text{ with } \lambda_*^{-1}(\mathfrak{g}) v_2 = 0\},$$

where $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n-2)} \mathfrak{h}$ is the orthogonal part of $\mathfrak{h}$. In view of Proposition 3.3, Theorem 3.4 and Proposition 6.1 this shows, that the orthogonal part $\mathfrak{g}$ of the holonomy algebra $\mathfrak{hol}(M,g)$ of a Lorentzian manifold with parallel spinor (together with its representation) coincides with the holonomy representation of a Riemannian manifold with parallel spinors. By Theorem 6.1, this representation splits into a trivial part and irreducible factors, which can be the standard representations of $\mathfrak{su}(m)$, $\mathfrak{sp}(k)$, $\mathfrak{g}_2$ or $\mathfrak{spin}(7)$. These Lie algebras have trivial center. In particular, $\mathfrak{hol}(M,g)$ can not be of type 4. We obtain finally

Theorem 6.2 ([L03]) *Let (M,g) be an indecomposable, simply connected Lorentzian manifold with non-trivial parallel spinor. Then the holonomy group is*

$$\mathrm{Hol}(M,g) = G \ltimes \mathbb{R}^{n-2},$$

where $G \subset \mathrm{SO}(n-2)$ is a product of Lie groups of the form $\{1\} \subset \mathrm{SO}(n_0)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, G_2 or $\mathrm{Spin}(7)$ and the representation of G on $\mathbb{R}^{n-2}$ is the direct sum of the standard representations of these groups.

The calculation of the spinor derivative of a general plane-fronted wave (11) shows easily, that such waves admit parallel spinors if and only if the Riemannian manifold

(F,h) admits such, and the number of independent parallel spinors on (M,g) is the same as on (F,h).

R. Bryant discussed local normal forms for pseudo-Riemannian metrics with parallel spinors in small dimension $n \leq 11$ (cf. [Bry00]). For the special case of abelian holonomy group $\mathrm{Hol}^0(M,g) = \mathbb{R}^{n-2}$ we know already the local normal form of such a metric. g is locally isometric to $(\mathbb{R}^n, g_H)$ with

$$g_H = 2dvdu + H\,du^2 + \sum_{i=1}^{n-2} dx_i^2,$$

where $H = H(x_1,\dots,x_{n-2},u)$ is an arbitrary smooth function. In view of Theorem 6.2 or Proposition 6.4 this is the only possible normal form for indecomposable Lorentzian manifolds with parallel spinors in dimension $n \leq 5$. For local metrics in dimension $6 \leq n \leq 11$ we refer to [Bry00], [F00], [F99], [H04], [BCH09], [BCH08], [CFH09] and the references therein.
We will address here to a global problem, namely to the question, whether one can realize the holonomy groups $G \ltimes \mathbb{R}^{n-2}$ which allow a parallel spinor by a globally hyperbolic manifold *with complete Cauchy surface*. In [BM08] we proved:

Theorem 6.3 *Any Lorentzian holonomy group of the form*

$$G \ltimes \mathbb{R}^{n-2} \subset \mathrm{SO}(1,n-1),$$

where $G \subset \mathrm{SO}(n-2)$ *is a product of Lie groups of the form* $\{1\} \subset \mathrm{SO}(n_0)$*,* $\mathrm{SU}(m)$*,* $\mathrm{Sp}(k)$*,* G_2 *or* $\mathrm{Spin}(7)$ *with its standard representations, can be realized by a globally hyperbolic Lorentzian manifold* (M^n,g) *with a complete Cauchy surface and a non-trivial parallel spinor.*

For the proof we use the characterization (13) of globally hyperbolic manifolds by Bernal and Sanchez and ideas from the paper [BGM05] of C. Bär, P. Gauduchon and A. Moroianu, who studied the spin geometry of generalized pseudo-Riemannian cylinders. First, we consider a special kind of spinor fields. Let (M_0,g_0) be a Riemannian spin manifold with a Codazzi tensor A, i.e., with a symmetric (1,1)-tensor field satisfying

$$(\nabla^{g_0}_X A)(Y) = (\nabla^{g_0}_Y A)(X) \qquad \text{for all vector fields } X, Y.$$

A spinor field φ on (M_0,g_0) is called *A-Codazzi spinor* if

$$\nabla^S_X \varphi = iA(X)\cdot\varphi \qquad \text{for all vector fields } X. \tag{14}$$

If A is uniformally bounded, we denote by $\mu_+(A)$ the supremum of the positive eigenvalues of A or zero if all eigenvalues are non-positive, and by $\mu_-(A)$ the infimum of the negative eigenvalues of A or zero if all eigenvalues are non-negative.

Proposition 6.5 *Let* (M_0,g_0) *be a complete Riemannian spin manifold with a uniformally bounded Codazzi tensor A and a non-trivial A-Codazzi spinor. Then the Lorentzian cylinder*

$$C := I \times M_0 \,,\; g_C := -dt^2 + (1-2tA)^* g_0,$$

with the interval $I = ((2\mu_-(A))^{-1}, (2\mu_+(A))^{-1})$ *is globally hyperbolic with complete Cauchy surface and with a parallel spinor.*

In order to obtain such cylinders, we have to ensure the existence of Codazzi spinors (14). Using our classification of Riemannian manifolds with imaginary Killing spinors ([B89]), we obtain:

Proposition 6.6 *Let (M_0, g_0) be a complete Riemannian manifold with an A-Codazzi spinor and let all eigenvalues of the Codazzi tensor A be uniformally bounded away from zero. Then (M_0, g_0) is isometric to*

$$(\mathbb{R} \times F \ , \ (A^{-1})^*(ds^2 + e^{-4s} g_F)),$$

where (F, h) is a complete Riemannian manifold with parallel spinors, and A^{-1} is a Codazzi-tensor on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$.
Vice versa, let (F, h) be a complete Riemannian manifold with parallel spinors and a Codazzi tensor T which has eigenvalues uniformally bounded from below. Then there is a Codazzi tensor B on the warped product

$$M_0 = \mathbb{R} \times F \ , \ g_{wp} = ds^2 + e^{-4s} g_F$$

with eigenvalues uniformally bounded away from zero. Moreover, B^{-1} is a Codazzi tensor on $(M_0, g_0 := (B^{-1})^* g_{wp})$, the Riemannian manifold (M_0, g_0) is complete and has B^{-1}–Codazzi spinors.

A Codazzi tensor B on a warped product

$$M_0 = \mathbb{R} \times F \ , \ g_{wp} = ds^2 + f(s)^2 g_F$$

with properties mentioned in Proposition 6.6 can be constructed from a Codazzi tensor T on (F, g_F) in the following way. We set

$$B := \begin{pmatrix} b \cdot \mathrm{Id} & 0 \\ 0 & E \end{pmatrix}$$

with respect to the decomposition $TM = \mathbb{R} \oplus TF$, where b is a function depending only on s and E is given by

$$E(s) = \frac{1}{f(s)} \Big(T + \int_0^s b(\sigma) \dot{f}(\sigma) d\sigma \cdot \mathrm{Id}_F \Big).$$

This yields a construction principle for globally hyperbolic manifolds with complete Cauchy surface and special holonomy.

Proposition 6.7 ([BM08]) *Let (F, g_F) be a complete Riemannian manifold with parallel spinors and a Codazzi tensor T with eigenvalues bounded from below. Then there are Codazzi tensors B on $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$ with eigenvalues uniformally bounded away from zero. Let*

$$C(F, B) := I \times \mathbb{R} \times F \ , \ g_C := -dt^2 + (B - 2t)^*(ds^2 + e^{-4s} g_F).$$

Then

1. *(C, g_C) is globally hyperbolic with a complete Cauchy surface, it admits a parallel light-like vector field as well as a parallel spinor.*

2. *If (F,h) has a flat factor, then $C(F,B)$ is decomposable.*
3. *If (F,h) is (locally) a product of irreducible factors, then $C(F,B)$ is weakly irreducible and*

$$\mathrm{Hol}^0_{(0,0,x)}(C,g_C) = (B^{-1}\circ \mathrm{Hol}^0_x(F,g_F)\circ B) \ltimes \mathbb{R}^{\dim F}.$$

Our construction is based on the existence of Codazzi tensors on Riemannian manifolds with parallel spinors. Let us finally discuss some examples for that.

Example 1. On the flat space $\mathbb{R}^k$ the endomorphism $T_h^{\mathbb{R}^k}$,

$$T_h^{\mathbb{R}^k}(X) := \nabla^{\mathbb{R}^k}_X(\mathrm{grad}(h)) = X(\partial_1 h,\dots,\partial_k h),$$

is a Codazzi tensor for any function h on $\mathbb{R}^k$, and every Codazzi tensor is of this form. In this case the cylinder $C(F,B)$ is flat for any Codazzi tensor B on the warped product that is constructed out of T as described above.

Example 2. Let (F_1,g_{F_1}) be a complete simply connected irreducible Riemannian spin manifold with parallel spinors and (F,g_F) its Riemannian product with a flat $\mathbb{R}^k$. Then (F,g_F) is complete and has parallel spinors. Let B be a Codazzi tensor on the warped product $\mathbb{R}\times_{e^{-2s}} F$ constructed out of the Codazzi tensor $\lambda\,\mathrm{Id}_{F_1} + T_h^{\mathbb{R}^k}$ of F, where $T_h^{\mathbb{R}^k}$ is taken from Example 1. Then the cylinder $C(F,B)$ is globally hyperbolic with complete Cauchy surface, it is decomposable and has the holonomy group

$$\mathrm{Hol}(F_1,g_{F_1}) \ \ltimes\ \mathbb{R}^{\dim F_1}.$$

Example 3. Let us consider the metric cone

$$(F^{n-2},g_F) := (\mathbb{R}^+\times N, dr^2 + r^2 g_N),$$

where (N,g_N) is simply connected and a Riemannian Einstein-Sasaki manifold, a nearly Kähler manifold, a 3-Sasakian manifold or a 7-dimensional manifold with vector product. Then (F,g_F) is irreducible and has parallel spinors (but fails to be complete). Furthermore, $T := \nabla^F \partial_r$ is a Codazzi tensor on (F,g_F). The cylinder $C(F,B)$, where the Codazzi tensor B is constructed out of T as described above, has the holonomy group

$$\mathrm{Hol}(C,g_C) \simeq G \ltimes \mathbb{R}^{n-2},$$

where

$$G = \begin{cases} \mathrm{SU}((n-2)/2) & \text{if} \quad N \text{ is Einstein-Sasaki} \\ \mathrm{Sp}((n-2)/4) & \text{if} \quad N \text{ is 3-Sasakian} \\ \mathrm{G}_2 & \text{if} \quad N \text{ is nearly Kähler} \\ \mathrm{Spin}(7) & \text{if} \quad N \text{ 7-dimensional with vector product.} \end{cases}$$

Example 4. Let $(F,g_F) = (F_1,g_{F_1})\times\cdots\times(F_k,g_{F_k})$ be a Riemannian product of simply connected complete irreducible Riemannian manifolds with parallel spinors. Let T be the Codazzi tensor $T = \lambda_1\,\mathrm{Id}_{F_1}+\cdots+\lambda_k\,\mathrm{Id}_{F_k}$ and B constructed out of T as described above. Then $C(F,B)$ is globally hyperbolic with complete Cauchy surface, it is weakly irreducible and the holonomy group is isomorphic to

$$\big(\mathrm{Hol}(F_1,g_{F_1})\times\cdots\times \mathrm{Hol}(F_k,g_{F_k})\big) \ \ltimes\ \mathbb{R}^{\dim F}.$$

Example 5. Eguchi-Hansen space. Eguchi-Hansen spaces are complete, irreducible Riemannian 4-manifolds with holonomy $SU(2)$. They have 2 linearly independent parallel spinors. Any Codazzi tensor on a Eguchi-Hansen space has the form $T = \lambda \cdot \mathrm{Id}$ for a constant λ.

6.2 Einstein metrics

In the final section we want to discuss recent results concerning Lorentzian Einstein spaces with special holonomy. As a first example let us look at the general plane-fronted wave (11). The Ricci tensor of such metric is given by

$$Ric = Ric^F - \frac{1}{2}\Delta^F H \cdot du^2.$$

Hence, for any Ricci-flat Riemannian manifold (F,h) and any family of harmonic functions $H(\cdot,u)$ on F, the general plane-fronted wave (11) is Ricci-flat with special holonomy.

Now, let (M,g) be a Lorentzian Einstein-space with Einstein constant Λ:

$$Ric = \Lambda \cdot g.$$

We suppose, that (M,g) admits a 1-dimensional parallel light-like distribution, i.e., that the holonomy group is contained in $(\mathbb{R}^* \times O(n-2)) \ltimes \mathbb{R}^{n-2}$. First we discuss the possible holonomy groups for such Einstein metrics. After that we review some results concerning the local structure of such metrics. We follow the papers of G. Gibbons and N. Pope ([GiP08], [Gi09]) as well as the results of T. Leistner and A. Galaev ([GL08], [GL10], [G10c].

Recall, that an irreducible Riemannian manifold with holonomy algebra different from $\mathfrak{so}(n)$ and $\mathfrak{u}(n/2)$ is Einstein. The determination of the possible holonomy algebras for Lorentzian Einstein spaces is based on a detailed study of the space of curvature endomorphisms $\mathscr{K}(\mathfrak{h})$ of a weakly irreducible subalgebra $\mathfrak{h} \subset (\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes \mathbb{R}^{n-2}$, which one can find in the papers of A. Galaev ([G05], [G10c], [G10a]).

Theorem 6.4 *Let (M,g) be a weakly-irreducible, but non-irreducible Lorentzian Einstein manifold. Then its holonomy algebra $\mathfrak{hol}(M,g)$ is of type 1 or type 2.*

1. If (M,g) is Ricci-flat, then the holonomy algebra is either $(\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$ and in the decomposition (1) of the orthogonal part $\mathfrak{g}$ at least one of the ideals $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}(n_i/2)$, $\mathfrak{sp}(n_i/4) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra, or the holonomy algebra is $\mathfrak{g} \ltimes \mathbb{R}^{n-2}$ and each ideal $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$ in the decomposition of $\mathfrak{g}$ coincides with one of the Lie-algebras $\mathfrak{so}(n_i)$, $\mathfrak{su}(n_i/2)$, $\mathfrak{sp}(n_i/4)$, $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$.

2. If (M,g) is an Einstein space with non-zero Einstein constant Λ, then the holonomy algebra is $(\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$, $\mathfrak{g}$ has no trivial invariant subspace and each ideal $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$ in the decomposition of $\mathfrak{g}$ is one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}(n_i/2)$, $\mathfrak{sp}(n_i/4) \oplus \mathfrak{sp}(1)$ or a symmetric Berger algebra.

We remark that contrary to the Riemannian situation, any of the holonomy algebras in Theorem 6.4 can be realized also by non-Einstein metrics.

Now, let us look at the local structure of a Lorentzian Einstein metric with a parallel light-like line and dimension at least 4. As we know from section 4, locally such metric is a Walker metric. Around any point $p \in M$ there are coordinates $(U,(v,x_1,\dots,x_{n-2},u))$ such that

$$g_{|U} = 2dvdu + H\,du^2 + 2A(u)\odot du + h(u), \tag{15}$$

where $h(u) = h_{ij}(x_1,\dots,x_{n-2},u)dx_idx_j$ is an u-depending family of Riemannian metrics, H is a smooth function on U and $A(u) = A_i(x_1,\dots,x_{n-2},u)dx_i$ is a u-depending family of 1-forms on U. Of course, the Einstein condition imposes conditions on the data A, H and h in the Walker metric. These conditions were derived by Gibbons and Pope in [GiP08].

Theorem 6.5 ([GiP08]) *Let (M,g) be a Lorentzian manifold with a parallel light-like line and assume that (M,g) is Einstein with Einstein constant Λ. Then the function H in the Walker metric (15) has the form*

$$H = \Lambda v^2 + vH_1 + H_0, \tag{16}$$

where H_1 and H_0 are smooth functions on U which do not depend on v, and H_0, H_1, $A(u)$ and $h(u)$ satisfy the following system of differential equations:

$$\begin{aligned}
\Delta H_0 - \frac{1}{2}F^{ij}F_{ij} - 2A^i\partial_iH_1 - H_1\nabla^iA_i + 2\Lambda A^iA_i - 2\nabla^i\dot{A}_i & & \\
+\frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 &=& 0,\\
\nabla^jF_{ij} + \partial_iH_1 - 2\Lambda A_i + \nabla^j\dot{h}_{ij} - \partial_i(h^{jk}\dot{h}_{jk}) &=& 0,\\
\Delta H_1 - 2\Lambda\nabla^iA_i + \Lambda h^{ij}\dot{h}_{ij} &=& 0,\\
Ric_{ij} &=& \lambda h_{ij}.
\end{aligned}$$

Hereby i,j,k run from 1 to $n-2$, the dot denotes the derivative with respect to u and ∂_i the derivative with respect to x_i, Δ is the Laplace-Beltrami operator for the metrics $h(u)$ and $F_{ij} = \partial_iA_j - \partial_jA_i$ are the coefficients of the differential of the 1-form $A(u)$. Conversely, any Walker metric (15) satisfying these equations is an Einstein metric with Einstein constant Λ.

In [GL10] Galaev and Leistner simplified this system of equations. They proved that one can always find Walker coordinates (15) with $A(u) = 0$. Moreover, using the special form of the curvature endomorphisms $\mathscr{K}((\mathbb{R}\oplus\mathfrak{so}(n-2))\ltimes\mathbb{R}^{n-2})$ and the condition (16), they were able to show that for an Einstein manifold there exist Walker coordinates with $A(u) = 0$ and $H_0 = 0$, and furthermore, if $\Lambda \neq 0$ one can choose Walker coordinates with $A(u) = 0$ and $H_1 = 0$. If the Einstein manifold admits not only a parallel light-like line, but a parallel light-like vector field, then by Theorem 6.4 the Einstein constant Λ is zero. In [GiP08] one can find a lot of concrete Lorentzian Einstein metrics with a parallel light-like line, which are of physical relevance (time-dependent multi-center solutions). The case of 4-dimensional Einstein spaces was previously discussed for example in [KG61a], [KG61b], [GT01]. More concrete solutions in dimension 4 are obtained in [G10b].

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