Russian Math. Surveys **0**:0 1–000 *Uspekhi Mat. Nauk* **0**:0 55–108
DOI

Holonomy groups of Lorentzian manifolds

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Abstract. This paper contains a survey of recent results on classification of the connected holonomy groups of Lorentzian manifolds. A simplification of the construction of Lorentzian metrics with all possible connected holonomy groups is obtained. The Einstein equation, Lorentzian manifolds with parallel and recurrent spinor fields, conformally flat Walker metrics, and the classification of 2-symmetric Lorentzian manifolds are considered as applications.

Bibliography: 123 titles.

Keywords: Lorentzian manifold, holonomy group, holonomy algebra, Walker manifold, Einstein equation, recurrent spinor field, conformally flat manifold, 2-symmetric Lorentzian manifold.

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AMS 2010 *Mathematics Subject Classification.* Primary 53C29, 53C50, 53B30.

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1. Introduction

The notion of holonomy group was introduced for the first time in the works [42] and [44] of É. Cartan. In [43] he used holonomy groups to obtain a classification of Riemannian symmetric spaces. The holonomy group of a pseudo-Riemannian manifold is a Lie subgroup of the Lie group of pseudo-orthogonal transformations of the tangent space at a point of the manifold and consists of parallel displacements along piecewise smooth loops at this point. Usually one considers the connected holonomy group, that is, the connected component of the identity of the holonomy group. For its definition one must consider parallel displacements along contractible loops. The Lie algebra corresponding to the holonomy group is called the holonomy algebra. The holonomy group of a pseudo-Riemannian manifold is an invariant of the corresponding Levi-Civita connection; it gives information about the curvature tensor and about parallel sections of the vector bundles associated to the manifold, such as the tensor bundle or the spinor bundle.

An important result is the Berger classification of connected irreducible holonomy groups of Riemannian manifolds [23]. It turns out that the connected holonomy group of an n-dimensional indecomposable not locally symmetric Riemannian manifold is contained in the following list: $\mathrm{SO}(n)$; $\mathrm{U}(m)$, $\mathrm{SU}(m)$ $(n=2m)$; $\mathrm{Sp}(m)$, $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ $(n=4m)$; $\mathrm{Spin}(7)$ $(n=8)$; G_2 $(n=7)$. Berger just obtained a list of possible holonomy groups, and the problem arose of showing that there exists a manifold with each of these holonomy groups. In particular, this led to the well-known Calabi–Yau theorem [123]. Only in 1987 did Bryant [38] construct examples of Riemannian manifolds with the holonomy groups $\mathrm{Spin}(7)$ and G_2. Thus, the solution of this problem required more then thirty years. The de Rham decomposition theorem [48] reduces the classification problem for the connected holonomy groups of Riemannian manifolds to the case of irreducible holonomy groups.

Indecomposable Riemannian manifolds with special holonomy groups (that is, not $\mathrm{SO}(n)$) have important geometric properties. Manifolds with the most of these holonomy groups are Einstein or Ricci-flat and admit parallel spinor fields. These properties ensured that Riemannian manifolds with special holonomy groups found applications in theoretical physics (in string theory, supersymmetry theory, and M-theory) [25], [45], [79], [88], [89], [103]. In this connection a large number of works have appeared over the past 20 years in which constructions of complete and compact Riemannian manifolds with special holonomy groups were described. We cite only some of them: [18], [20], [47], [51], [88], [89]. It is important to note that in string theory and M-theory it is assumed that our space is locally a product

$$\mathbb{R}^{1,3}\times M \tag{1.1}$$

of the Minkowski apace $\mathbb{R}^{1,3}$ and some compact Riemannian manifold M of dimension 6, 7, or 8 and with holonomy group $\mathrm{SU}(3)$, G_2, or $\mathrm{Spin}(7)$, respectively. Parallel spinor fields on M define supersymmetries.

It is natural to consider the classification problem for connected holonomy groups of pseudo-Riemannian manifolds, and first and foremost, of all Lorentzian manifolds.

There is the Berger classification of connected irreducible holonomy groups of pseudo-Riemannian manifolds [23]. However, in the case of pseudo-Riemannian manifolds it is not enough to consider only irreducible holonomy groups. The Wu decomposition theorem [122] lets us restrict consideration to the connected weakly irreducible holonomy groups. A weakly irreducible holonomy group does not preserve any proper non-degenerate vector subspace of the tangent space. Such a holonomy group can preserve a degenerate subspace of the tangent space, in which case the holonomy group is not reductive. And therein lies the main problem.

For a long time there were only results on the holonomy groups of four-dimensional Lorentzian manifolds: [10], [81], [87], [91], [92], [102], [109]. In these papers a classification of connected holonomy groups is obtained, and the connections between it and the Einstein equation, the Petrov classification of gravitational fields [108], and other problems in general relativity are considered.

In 1993, Bérard-Bergery and Ikemakhen made the first step towards classification of the connected holonomy groups of Lorentzian manifolds of arbitrary dimension [21]. We describe all subsequent steps of the classification and its consequences.

In § 2 of the present paper we give definitions and some known results about the holonomy groups of Riemannian and pseudo-Riemannian manifolds.

In § 3 we start to study the holonomy algebras $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ of Lorentzian manifolds (M, g) of dimension $n + 2 \geqslant 4$. The Wu theorem lets us assume that the holonomy algebra is weakly irreducible. If $\mathfrak{g} \neq \mathfrak{so}(1, n+1)$, then $\mathfrak{g}$ preserves some isotropic line of the tangent space and is contained in the maximal subalgebra $\mathfrak{sim}(n) \subset \mathfrak{so}(1, n+1)$ preserving this line. First of all, we give a geometric interpretation [57] of the classification by Bérard-Bergery and Ikemakhen [21] of weakly irreducible subalgebras of $\mathfrak{g} \subset \mathfrak{sim}(n)$. It turns out that these subalgebras are exhausted by the Lie algebras of transitive Lie groups of similarity transformations of the Euclidean space $\mathbb{R}^n$.

We next study the question as to which of the subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ obtained are the holonomy algebras of Lorentzian manifolds. First of all, it is necessary to classify the Berger subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$, which are the algebras spanned by the images of the elements of the space $\mathscr{R}(\mathfrak{g})$ of algebraic curvature tensors (tensors satisfying the first Bianchi identity) and are candidates for the holonomy algebras. In § 4 we describe the structure of the spaces $\mathscr{R}(\mathfrak{g})$ of curvature tensors for the subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ [55] and reduce the classification problem for Berger algebras to the classification problem for weak Berger algebras $\mathfrak{h} \subset \mathfrak{so}(n)$, which are the algebras spanned by the images of the elements of the space $\mathscr{P}(\mathfrak{h})$ consisting of the linear maps from $\mathbb{R}^n$ to $\mathfrak{h}$ that satisfy a certain identity. Next we find the curvature tensor of Walker manifolds, that is, manifolds with the holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$.

In § 5 the results of computations of the spaces $\mathscr{P}(\mathfrak{h})$ in [59] are presented. This gives the complete structure of the spaces of curvature tensors for the holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$. The space $\mathscr{P}(\mathfrak{h})$ appeared as the space of values of a certain component of the curvature tensor of a Lorentzian manifold. Later it turned out that this space also contains a component of the curvature tensor of a Riemannian supermanifold [63].

Leistner [100] classified weak Berger algebras, showing in a far from trivial way that they are exhausted by the holonomy algebras of Riemannian spaces. The natural problem of finding a simple direct proof of this fact arises. In § 6 we present such a proof from [68] for the case of semisimple not simple irreducible Lie algebras $\mathfrak{h} \subset \mathfrak{so}(n)$. Leistner's theorem yields a classification of Berger subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$.

In § 7 we prove that all Berger algebras can be realized as the holonomy algebras of Lorentzian manifolds, and we greatly simplify the constructions of the metrics in [56]. By this we complete the classification of holonomy algebras of Lorentzian manifolds.

The problem arises of constructing examples of Lorentzian manifolds with various holonomy groups and additional global geometric properties. In [17] and [19] there are constructions for globally hyperbolic Lorentzian manifolds with certain classes of holonomy groups. Global hyperbolicity is a strong causality condition in Lorentzian geometry that generalizes the concept of completeness in Riemannian geometry. In [95] some constructions using the Kaluza–Klein idea were presented. In the papers [16], [97], and [101] various global geometric properties of Lorentzian manifolds with different holonomy groups were studied. Holonomy groups were discussed in the recent survey [105] on global Lorentzian geometry. In [16] Lorentzian manifolds with disconnected holonomy groups were considered, and some examples were given. In [70] and [71] we gave algorithms for computing the holonomy algebra of an arbitrary Lorentzian manifold.

We next consider some applications of the classification obtained.

In § 8 we study the connection between holonomy algebras and the Einstein equation. This topic is motivated by the paper [76] by the theoretical physicists Gibbons and Pope, in which the problem of finding the Einstein metrics with holonomy algebras in $\mathfrak{sim}(n)$ was proposed, examples were considered, and their physical interpretation was given. We find the holonomy algebras of Einstein Lorentzian manifolds [60], [61]. Next, we show that on each Walker manifold there exist special coordinates enabling one to essentially simplify the Einstein equation [74]. Examples of Einstein metrics from [60] and [62] are given.

In § 9 results about Riemannian and Lorentzian manifolds admitting recurrent spinor fields [67] are presented. Recurrent spinor fields generalize parallel spinor fields. Simply connected Riemannian manifolds with parallel spinor fields were classified in [121] in terms of their holonomy groups. A similar problem for Lorentzian manifolds was considered in [40], [52], and solved in [98], [99]. The connection between the holonomy groups of Lorentzian manifolds and the solutions of certain other spinor equations was discussed in [12], [13], [17], and the physical literature cited below.

In § 10 a local classification is obtained for conformally flat Lorentzian manifolds with special holonomy groups [66]. The corresponding local metrics are certain extensions of Riemannian spaces of constant sectional curvature to Walker metrics.

It is noted that earlier there was the problem of finding examples of such metrics in dimension 4 [75], [81].

In § 11 we obtain a classification of 2-symmetric Lorentzian manifolds, that is, manifolds satisfying the conditions $\nabla^2 R = 0$, $\nabla R \neq 0$. We discuss and simplify the proof of this result in [5], demonstrating applications of the theory of holonomy groups. The classification problem for 2-symmetric manifolds has been studied also in [28], [29], [90], [112].

Lorentzian manifolds with weakly irreducible not irreducible holonomy groups admit parallel distributions of isotropic lines; such manifolds are also called Walker manifolds [37], [120]. These manifolds are studied in geometric and physical literature. In [35], [36], and [77] the hope is expressed that the Lorentzian manifolds with special holonomy groups will find applications in theoretical physics, for example, in M-theory and string theory. It is proposed to replace the manifold (1.1) by an indecomposable Lorentzian manifold with an appropriate holonomy group. In connection with the 11-dimensional supergravity theory there have been recent physics papers with studies of 11-dimensional Lorentzian manifolds admitting spinor fields satisfying certain equations, and holonomy groups were used [11], [53], [113]. We mention also [45], [46], [78]. All this indicates the importance of studying the holonomy groups of Lorentzian manifolds and the related geometric structures.

In the case of pseudo-Riemannian manifolds with signatures different from Riemannian and Lorentzian there is not yet a classification of holonomy groups. There are only some partial results ([22], [26], [27], [30], [58], [65], [69], [73], [85]).

Finally, we mention some other results on holonomy groups. Consideration of the cone over a Riemannian manifold enables one to obtain Riemannian metrics with special holonomy groups and to interpret Killing spinor fields as parallel spinor fields on the cone [34]. Furthermore, the holonomy groups of the cones over pseudo-Riemannian manifolds, and in particular over Lorentzian manifolds, are studied in the paper [4]. There are results about irreducible holonomy groups of linear torsion-free connections in [9], [39], [104], and [111]. Holonomy groups are defined also for manifolds with conformal metrics. In particular, for these groups it is possible to decide whether there are Einstein metrics in the conformal class [14]. The notion of holonomy group is used also for superconnections on supermanifolds [1], [63].

The author is thankful to D. V Alekseevsky for useful discussions and suggestions.

2. Holonomy groups and algebras: definitions and facts

In this section we recall some definitions and known facts about holonomy groups of pseudo-Riemannian manifolds ([25], [88], [89], [94]). All manifolds are assumed to be connected.

2.1. Holonomy groups of connections in vector bundles. Let M be a smooth manifold and E a vector bundle over M with a connection ∇. The connection determines a parallel displacement: for any piecewise smooth curve $\gamma\colon [a,b] \subset \mathbb{R} \to M$ a vector space isomorphism

$$\tau_\gamma\colon E_{\gamma(a)} \to E_{\gamma(b)}$$

is defined. For a fixed point $x \in M$ the holonomy group G_x of the connection ∇ at x is defined as the group consisting of parallel displacements along all piecewise smooth loops at the point x. If we consider only null-homotopic loops, then we get the restricted holonomy group G^0_x. If the manifold M is simply connected, then $G^0_x = G_x$. It is known that G_x is a Lie subgroup of the Lie group $\mathrm{GL}(E_x)$, and the group G^0_x is the connected component of the identity of the Lie group G_x. Let $\mathfrak{g}_x \subset \mathfrak{gl}(E_x)$ be the corresponding Lie algebra, which is called the holonomy algebra of the connection ∇ at the point x. The holonomy groups at different points of a connected manifold are isomorphic, and one can speak about the holonomy group $G \subset \mathrm{GL}(m, \mathbb{R})$ or about the holonomy algebra $\mathfrak{g} \subset \mathfrak{gl}(m, \mathbb{R})$ of the connection ∇ (here m is the rank of the vector bundle E). In the case of a simply connected manifold, the holonomy algebra determines the holonomy group uniquely.

Recall that a section $X \in \Gamma(E)$ is said to be parallel if $\nabla X = 0$. This is equivalent to the condition that for any piecewise smooth curve $\gamma : [a, b] \to M$ we have $\tau_\gamma X_{\gamma(a)} = X_{\gamma(b)}$. Similarly, a subbundle $F \subset E$ is said to be parallel if for any section X of the subbundle F and for any vector field Y on M, the section $\nabla_Y X$ again belongs to F. This is equivalent to the property that for any piecewise smooth curve $\gamma\colon [a, b] \to M$ we have $\tau_\gamma F_{\gamma(a)} = F_{\gamma(b)}$.

The following fundamental principle shows the importance of holonomy groups.

Theorem 1. *There is a one-to-one correspondence between parallel sections X of the bundle E and vectors $X_x \in E_x$ invariant with respect to G_x.*

Let us describe this correspondence. For a parallel section X take the value X_x at a point $x \in M$. Since X is invariant under parallel displacements, the vector X_x is invariant under the holonomy group. Conversely, for a given vector X_x we define the section X. For any point $y \in M$ put $X_y = \tau_\gamma X_x$, where γ is any curve beginning at x and ending at y. The value X_y does not depend on the choice of the curve γ.

A similar result holds for subbundles.

Theorem 2. *There is a one-to-one correspondence between parallel subbundles $F \subset E$ and vector subspaces $F_x \subset E_x$ invariant with respect to G_x.*

The next theorem, proved by Ambrose and Singer [8], shows the relation between the holonomy algebra and the curvature tensor R of the connection ∇.

Theorem 3. *Let $x \in M$. The Lie algebra $\mathfrak{g}_x$ is spanned by the operators of the form*

$$\tau_\gamma^{-1} \circ R_y(X, Y) \circ \tau_\gamma \in \mathfrak{gl}(E_x),$$

where γ is an arbitrary piecewise smooth curve beginning at x and ending at a point $y \in M$, and $Y, Z \in T_y M$.

2.2. Holonomy groups of pseudo-Riemannian manifolds. Let us consider pseudo-Riemannian manifolds. Recall that a pseudo-Riemannian manifold with signature (r, s) is a smooth manifold M equipped with a smooth field g of symmetric non-degenerate bilinear forms with signature (r, s) (r is the number of minuses) at each point. If $r = 0$, then such a manifold is called a Riemannian manifold. If $r = 1$, then (M, g) is called a Lorentzian manifold. In this case we assume for convenience that $s = n + 1$, $n \geqslant 0$.

On the tangent bundle TM of a pseudo-Riemannian manifold M one has the canonical Levi-Civita connection ∇ defined by the following two conditions: the field of forms g is parallel ($\nabla g = 0$) and the torsion is zero ($\mathrm{Tor} = 0$). Denote by $\mathrm{O}(T_xM, g_x)$ the group of linear transformation of T_xM preserving the form g_x. Since the metric g is parallel, $G_x \subset \mathrm{O}(T_xM, g_x)$. The tangent space (T_xM, g_x) can be identified with the pseudo-Euclidean space $\mathbb{R}^{r,s}$. The metric of this space we denote by g. Then we can identify the holonomy group G_x with a Lie subgroup of $\mathrm{O}(r, s)$, and the holonomy algebra $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{so}(r, s)$.

The connection ∇ is in a natural way extendable to a connection in the tensor bundle $\otimes^p_q TM$, and the holonomy group of this connection coincides with the natural representation of the group G_x on the tensor space $\otimes^p_q T_xM$. The following statement is implied by Theorem 1.

Theorem 4. *There is a one-to-one correspondence between parallel tensor fields A of type (p, q) and tensors $A_x \in \otimes^p_q T_xM$ invariant with respect to G_x.*

Thus if we know the holonomy group of a manifold, then the geometric problem of finding the parallel tensor fields on the manifold can be reduced to the simpler algebraic problem of finding the invariants of the holonomy group. Let us consider several examples illustrating this principle.

Recall that a pseudo-Riemannian manifold (M, g) is said to be flat if (M, g) admits local parallel fields of frames. We get that (M, g) is flat if and only if $G^0 = \{\mathrm{id}\}$ (or $\mathfrak{g} = \{0\}$). Moreover, from the Ambrose–Singer theorem it follows that the last equality is equivalent to the nullity of the curvature tensor.

A pseudo-Riemannian manifold (M, g) is said to be *pseudo-Kählerian* if on M there exists a parallel field of endomorphisms J with the properties $J^2 = -\mathrm{id}$ and $g(JX, Y) + g(X, JY) = 0$ for all vector fields X and Y on M. It is obvious that a pseudo-Riemannian manifold (M, g) with signature $(2r, 2s)$ is pseudo-Kählerian if and only if $G \subset \mathrm{U}(r, s)$.

For an arbitrary subalgebra $\mathfrak{g} \subset \mathfrak{so}(r, s)$ let

$$\mathscr{R}(\mathfrak{g}) = \bigl\{R \in \mathrm{Hom}(\wedge^2\mathbb{R}^{r,s}, \mathfrak{g}) \mid R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \text{ for all } X, Y, Z \in \mathbb{R}^{r,s}\bigr\}.$$

The space $\mathscr{R}(\mathfrak{g})$ is called the space of curvature tensors of type $\mathfrak{g}$. We denote by $L(\mathscr{R}(\mathfrak{g}))$ the vector subspace of $\mathfrak{g}$ spanned by the elements of the form $R(X, Y)$ for all $R \in \mathscr{R}(\mathfrak{g})$, $X, Y \in \mathbb{R}^{r,s}$. From the Ambrose–Singer theorem and the first Bianchi identity it follows that if $\mathfrak{g}$ is the holonomy algebra of a pseudo-Riemannian space (M, g) at a point $x \in M$, then $R_x \in \mathscr{R}(\mathfrak{g})$, that is, knowledge of the holonomy algebra enables one to get restrictions on the curvature tensor, a fact that will be used repeatedly below. Moreover, $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$. A subalgebra $\mathfrak{g} \subset \mathfrak{so}(r, s)$ is called a Berger algebra if $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$. It is natural to consider the Berger algebras as candidates for the holonomy algebras of pseudo-Riemannian manifolds. Each element $R \in \mathscr{R}(\mathfrak{so}(r, s))$ has the property

$$(R(X, Y)Z, W) = (R(Z, W)X, Y), \qquad X, Y, Z, W \in \mathbb{R}^{r,s}. \tag{2.1}$$

Theorem 3 does not give a good way to find the holonomy algebra. Sometimes it is possible to use the following theorem.

Theorem 5. *If the pseudo-Riemannian manifold (M,g) is analytic, then the holonomy algebra $\mathfrak{g}_x$ is generated by the following operators:*

$$R(X,Y)_x, \nabla_{Z_1}R(X,Y)_x, \nabla_{Z_2}\nabla_{Z_1}R(X,Y)_x, \ldots \in \mathfrak{so}(T_xM, g_x),$$

where $X, Y, Z_1, Z_2, \ldots \in T_xM$.

A subspace $U \subset \mathbb{R}^{r,s}$ is said to be non-degenerate if the restriction of the form g to this subspace is non-degenerate. A Lie subgroup $G \subset \mathrm{O}(r,s)$ (or a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$) is said to be irreducible if it does not preserve any proper vector subspace of $\mathbb{R}^{r,s}$, and G (or $\mathfrak{g}$) is said to be weakly irreducible if it does not preserve any proper non-degenerate vector subspace of $\mathbb{R}^{r,s}$.

It is clear that a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$ is irreducible (respectively, weakly irreducible) if and only if the corresponding connected Lie subgroup $G \subset \mathrm{SO}(r,s)$ is irreducible (respectively, weakly irreducible). If a subgroup $G \subset \mathrm{O}(r,s)$ is irreducible, then it is weakly irreducible. The converse holds only for positive- and negative-definite metrics g.

Let us consider two pseudo-Riemannian manifolds (M,g) and (N,h). For $x \in M$ and $y \in N$ let G_x and H_y be the corresponding holonomy groups. The direct product $M \times N$ of the manifolds is a pseudo-Riemannian manifold with respect to the metric $g+h$. A pseudo-Riemannian manifold is said to be (locally) indecomposable if it is not a (local) direct product of pseudo-Riemannian manifolds. Denote by $F_{(x,y)}$ the holonomy group of the manifold $M \times N$ at the point (x,y). Then $F_{(x,y)} = G_x \times H_y$. This statement has the following converse.

Theorem 6. *Let (M,g) be a pseudo-Riemannian manifold, and let $x \in M$. Suppose that the restricted holonomy group G^0_x is not weakly irreducible. Then the space T_xM admits an orthogonal decomposition (with respect to g_x) into a direct sum of non-degenerate linear subspaces:*

$$T_xM = E_0 \oplus E_1 \oplus \cdots \oplus E_t,$$

where G^0_x acts trivially on E_0, $G^0_x(E_i) \subset E_i$ $(i = 1, \ldots, t)$, and G^0_x acts weakly irreducibly on E_i $(i = 1, \ldots, t)$. There exist a flat pseudo-Riemannian submanifold $N_0 \subset M$ and locally indecomposable pseudo-Riemannian submanifolds $N_1, \ldots, N_t \subset M$ containing x such that $T_xN_i = E_i$ $(i = 0, \ldots, t)$, and there exist open subsets $U \subset M$, $U_i \subset N_i$ $(i = 0, \ldots, t)$ containing x such that

$$U = U_0 \times U_1 \times \cdots \times U_t, \qquad g\big|_{TU \times TU} = g\big|_{TU_0 \times TU_0} + g\big|_{TU_1 \times TU_1} + \cdots + g\big|_{TU_t \times TU_t}.$$

Moreover, there exists a decomposition

$$G^0_x = \{\mathrm{id}\} \times H_1 \times \cdots \times H_t,$$

where $H_i = G^0_x\big|_{E_i}$ are normal Lie subgroups of G^0_x $(i = 1, \ldots, t)$.

Furthermore, if M is simply connected and complete, then there exists a global decomposition

$$M = N_0 \times N_1 \times \cdots \times N_t.$$

A local statement of this theorem for the case of Riemannian manifolds was proved by Borel and Lichnerowicz [31]. The global statement for the case of Riemannian manifolds was proved by de Rham [48]. The statement of the theorem for pseudo-Riemannian manifolds was proved by Wu [122].

In [70] algorithms are obtained for finding the de Rham decomposition for Riemannian manifolds and the Wu decomposition for Lorentzian manifolds, with the use of an analysis of parallel bilinear symmetric forms on the manifold.

From Theorem 6 it follows that a pseudo-Riemannian manifold is locally indecomposable if and only if its restricted holonomy group is weakly irreducible.

It is important to note that the Lie algebras of the Lie groups H_i in Theorem 6 are Berger algebras. The next theorem is the algebraic version of Theorem 6.

Theorem 7. *Let $\mathfrak{g} \subset \mathfrak{so}(p,q)$ be a Berger subalgebra that is not irreducible. Then there exist an orthogonal decomposition*

$$\mathbb{R}^{p,q} = V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

and a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

into a direct sum of ideals such that $\mathfrak{g}_i$ annihilates V_j for $i \neq j$ and $\mathfrak{g}_i \subset \mathfrak{so}(V_i)$ is a weakly irreducible Berger subalgebra.

2.3. Connected irreducible holonomy groups of Riemannian and pseudo-Riemannian manifolds. In the previous subsection we saw that the classification problem for subalgebras $\mathfrak{g} \subset \mathfrak{so}(r,s)$ with the property $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$ can be reduced to the classification problem for weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{so}(r,s)$ with this property. For a subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$, weak irreducibility is equivalent to irreducibility. Recall that a pseudo-Riemannian manifold (M,g) is said to be locally symmetric if its curvature tensor satisfies the equality $\nabla R = 0$. For any locally symmetric Riemannian manifold there exists a simply connected Riemannian manifold with the same restricted holonomy group. Simply connected Riemannian symmetric manifolds were classified by É. Cartan [25], [43], [82]. If the holonomy group of such a space is irreducible, then it coincides with the isotropy subgroup. Thus, connected irreducible holonomy groups of locally symmetric Riemannian manifolds are known.

It is important to note that there is a one-to-one correspondence between simply connected indecomposable symmetric Riemannian manifolds (M,g) and simple $\mathbb{Z}_2$-graded Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ such that $\mathfrak{h} \subset \mathfrak{so}(n)$. The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ coincides with the holonomy algebra of the manifold (M,g). The space (M,g) can be reconstructed using its holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ and the value $R \in \mathscr{R}(\mathfrak{h})$ of the curvature tensor of the space (M,g) at some point. To this end, let us define a Lie algebra structure on the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ in the following way:

$$[A,B] = [A,B]_{\mathfrak{h}}, \quad [A,X] = AX, \quad [X,Y] = R(X,Y), \qquad A,B \in \mathfrak{h}, \quad X,Y \in \mathbb{R}^n.$$

Then, $M = G/H$, where G is a simply connected Lie group with the Lie algebra $\mathfrak{g}$, and $H \subset G$ is the connected Lie subgroup corresponding to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

In 1955 Berger obtained a list of possible connected irreducible holonomy groups of Riemannian manifolds [23].

Theorem 8. *If $G \subset \mathrm{SO}(n)$ is a connected Lie subgroup whose Lie algebra $\mathfrak{g} \subset \mathfrak{so}(n)$ satisfies the condition $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$, then either G is the holonomy group of a locally symmetric Riemannian space or G is one of the following groups*: $\mathrm{SO}(n)$; $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $n = 2m$; $\mathrm{Sp}(m)$, $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$, $n = 4m$; $\mathrm{Spin}(7)$, $n = 8$; G_2, $n = 7$.

The initial Berger list contained also the Lie group $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$. In [2] Alekseevsky showed that Riemannian manifolds with the holonomy group Spin(9) are locally symmetric. The list of possible connected irreducible holonomy groups of not locally symmetric Riemannian manifolds in Theorem 8 coincides with the list of connected Lie groups $G \subset \mathrm{SO}(n)$ acting transitively on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (if we exclude from the last list the Lie groups Spin(9) and $\mathrm{Sp}(m) \cdot T$, where T is the circle). After observing this, in 1962 Simons obtained in [114] a direct proof of Berger's result. A simpler and more geometric proof was found very recently by Olmos [107].

The proof of the Berger Theorem 8 is based on the classification of irreducible real linear representations of real compact Lie algebras. Each such representation can be obtained from the fundamental representations using tensor products and decompositions of the latter into irreducible components. Berger's proof reduces to a verification that such a representation (with several exceptions) cannot be the holonomy representation: from the Bianchi identity it follows that $\mathscr{R}(\mathfrak{g}) = \{0\}$ if the representation contains more then one tensor coefficient. It remains to investigate only the fundamental representations explicitly described by É. Cartan. By complicated computations it is possible to show that the Bianchi identity implies that either $\nabla R = 0$ or $R = 0$, aside from several exceptions indicated in Theorem 8.

Examples of Riemannian manifolds with the holonomy groups $\mathrm{U}(n/2)$, $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$, and $\mathrm{Sp}(n/4)\cdot\mathrm{Sp}(1)$ were constructed by Calabi, Yau, and Alekseevsky. In 1987 Bryant [40] constructed examples of Riemannian manifolds with the holonomy groups Spin(7) and G_2. This completes the classification of connected holonomy groups of Riemannian manifolds.

Let us describe the geometric structures on Riemannian manifolds with the holonomy groups in Theorem 8.

- $\mathrm{SO}(n)$: This is the holonomy group of generic Riemannian manifolds. There are no additional geometric structures related to the holonomy group on such manifolds.
- $\mathrm{U}(m)$ ($n = 2m$): Manifolds with this holonomy group are Kählerian, and on each of them there exists a parallel complex structure.
- $\mathrm{SU}(m)$ ($n = 2m$): Each manifold with this holonomy group is Kählerian and not Ricci-flat. They are called special Kählerian or Calabi–Yau manifolds.
- $\mathrm{Sp}(m)$ ($n = 4m$): On each manifold with this holonomy group there exists a parallel quaternionic structure, that is, parallel complex structures I, J, K connected by the relations $IJ = -JI = K$. These manifolds are said to be hyper-Kählerian.
- $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ ($n = 4m$): On each manifold with this holonomy group there exists a parallel 3-dimensional subbundle of the bundle of endomorphisms of the tangent spaces that is locally generated by the parallel quaternionic structure.
- $\mathrm{Spin}(7)$ ($n = 8$), G_2 ($n = 7$): Manifolds with these holonomy groups are Ricci-flat. On a manifold with the holonomy group Spin(7) there exists a parallel 4-form, and on each manifold with the holonomy group G_2 there exists a parallel 3-form.

Thus, indecomposable Riemannian manifolds with special (that is, different from $\mathrm{SO}(n)$) holonomy groups have important geometric properties. Because of these properties Riemannian manifolds with special holonomy groups have found applications in theoretical physics (in string theory and M-theory: [45], [79], [89]).

The spaces $\mathscr{R}(\mathfrak{g})$ for irreducible holonomy algebras $\mathfrak{g} \subset \mathfrak{so}(n)$ of Riemannian manifolds were computed by Alekseevsky [2]. For $R \in \mathscr{R}(\mathfrak{g})$ we define the corresponding Ricci tensor by

$$\mathrm{Ric}(R)(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y),$$

$X, Y \in \mathbb{R}^n$. The space $\mathscr{R}(\mathfrak{g})$ admits the following decomposition into a direct sum of $\mathfrak{g}$-modules:

$$\mathscr{R}(\mathfrak{g}) = \mathscr{R}_0(\mathfrak{g}) \oplus \mathscr{R}_1(\mathfrak{g}) \oplus \mathscr{R}'(\mathfrak{g}),$$

where $\mathscr{R}_0(\mathfrak{g})$ consists of the curvature tensors with zero Ricci tensors, $\mathscr{R}_1(\mathfrak{g})$ consists of the tensors annihilated by the Lie algebra $\mathfrak{g}$ (this space is either trivial or 1-dimensional), and $\mathscr{R}'(\mathfrak{g})$ is the complement of these two subspaces. If $\mathscr{R}(\mathfrak{g}) = \mathscr{R}_1(\mathfrak{g})$, then each Riemannian manifold with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(n)$ is locally symmetric. Such subalgebras $\mathfrak{g} \subset \mathfrak{so}(n)$ are called *symmetric Berger algebras*. The holonomy algebras of irreducible Riemannian symmetric spaces are exhausted by the algebras $\mathfrak{so}(n)$, $\mathfrak{u}(n/2)$, $\mathfrak{sp}(n/4) \oplus \mathfrak{sp}(1)$, and the symmetric Berger algebras $\mathfrak{g} \subset \mathfrak{so}(n)$. For the holonomy algebras $\mathfrak{su}(m)$, $\mathfrak{sp}(m)$, G_2, and $\mathfrak{spin}(7)$ we have $\mathscr{R}(\mathfrak{g}) = \mathscr{R}_0(\mathfrak{g})$, which shows that the manifolds with such holonomy algebras are Ricci-flat. Furthermore, for $\mathfrak{g} = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ we have $\mathscr{R}(\mathfrak{g}) = \mathscr{R}_0(\mathfrak{g}) \oplus \mathscr{R}_1(\mathfrak{g})$, and hence the corresponding manifolds are Einstein manifolds.

The next theorem, proved by Berger in 1955, gives a classification of the possible connected irreducible holonomy groups of pseudo-Riemannian manifolds [23].

Theorem 9. *If $G \subset \mathrm{SO}(r,s)$ is a connected irreducible Lie subgroup whose Lie algebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$ satisfies the condition $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$, then either G is the holonomy group of a locally symmetric pseudo-Riemannian space, or G is one of the following groups*: $\mathrm{SO}(r,s)$; $\mathrm{U}(p,q)$, $\mathrm{SU}(p,q)$, $r = 2p$, $s = 2q$; $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(p,q) \cdot \mathrm{Sp}(1)$, $r = 4p$, $s = 4q$; $\mathrm{SO}(r,\mathbb{C})$, $s = r$; $\mathrm{Sp}(p,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$, $r = s = 2p$; $\mathrm{Sp}(p,\mathbb{C}) \cdot \mathrm{SL}(2,\mathbb{C})$, $r = s = 4p$; $\mathrm{Spin}(7)$, $r = 0$, $s = 8$; $\mathrm{Spin}(4,3)$, $r = s = 4$; $\mathrm{Spin}(7)^{\mathbb{C}}$, $r = s = 8$; G_2, $r = 0$, $s = 7$; $G^*_{2(2)}$, $r = 4$, $s = 3$; $G_2^{\mathbb{C}}$, $r = s = 7$.

The proof of Theorem 9 uses the fact that a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$ satisfies the condition $L(\mathscr{R}(\mathfrak{g})) = \mathfrak{g}$ if and only if its complexification $\mathfrak{g}(\mathbb{C}) \subset \mathfrak{so}(r+s,\mathbb{C})$ satisfies the condition $L(\mathscr{R}(\mathfrak{g}(\mathbb{C}))) = \mathfrak{g}(\mathbb{C})$. In other words, Theorem 9 lists the connected real Lie groups whose Lie algebras exhaust the real forms of the complexifications of the Lie algebras for the Lie groups in Theorem 8.

In 1957 Berger [24] obtained the list of connected irreducible holonomy groups of pseudo-Riemannian symmetric spaces (we do not give this list here because it is too large).

3. Weakly irreducible subalgebras of $\mathfrak{so}(1, n+1)$

In this section we present the geometric interpretation in [57] of the classification of weakly irreducible subalgebras of $\mathfrak{so}(1, n+1)$ by Bérard-Bergery and Ikemakhen [21].

We proceed to the study of holonomy algebras of Lorentzian manifolds. Consider a connected Lorentzian manifold (M,g) of dimension $n+2\geqslant 4$. We identify the tangent space at some point of (M,g) with the Minkowski space $\mathbb{R}^{1,n+1}$. We denote by g the Minkowski metric on $\mathbb{R}^{1,n+1}$. Then the holonomy algebra $\mathfrak{g}$ of (M,g) at this point is identified with a subalgebra of the Lorentzian Lie algebra $\mathfrak{so}(1,n+1)$. By Theorem 6, (M,g) is not locally a product of pseudo-Riemannian manifolds if and only if its holonomy algebra $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ is weakly irreducible. Therefore, we will assume that $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ is weakly irreducible. If $\mathfrak{g}$ is irreducible, then $\mathfrak{g}=\mathfrak{so}(1,n+1)$. This follows from results of Berger. In fact, $\mathfrak{so}(1,n+1)$ does not contain any proper irreducible subalgebra (direct geometric proofs of this statement can be found in [50] and [33]). Thus, we can assume that $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ is weakly irreducible and not irreducible, and then $\mathfrak{g}$ preserves some degenerate subspace $U\subset\mathbb{R}^{1,n+1}$ and also the isotropic line $\ell=U\cap U^\perp\subset\mathbb{R}^{1,n+1}$. We fix an arbitrary isotropic vector $p\in\ell$, so that $\ell=\mathbb{R}p$. Let us fix some other isotropic vector q such that $g(p,q)=1$. The subspace $E\subset\mathbb{R}^{1,n+1}$ orthogonal to the vectors p and q is Euclidean, and we usually denote it by $\mathbb{R}^n$. Let $e_1,\dots,e_n$ be an orthogonal basis in $\mathbb{R}^n$. We get the Witt basis $p,e_1,\dots,e_n,q$ of the space $\mathbb{R}^{1,n+1}$.

Denote by $\mathfrak{so}(1,n+1)_{\mathbb{R}p}$ the maximal subalgebra of $\mathfrak{so}(1,n+1)$ preserving the isotropic line $\mathbb{R}p$. The Lie algebra $\mathfrak{so}(1,n+1)_{\mathbb{R}p}$ can be identified with the matrix Lie algebra

$$\mathfrak{so}(1,n+1)_{\mathbb{R}p}=\left\{\left.\begin{pmatrix}a & X^t & 0\\ 0 & A & -X\\ 0 & 0 & -a\end{pmatrix}\,\right|\, a\in\mathbb{R},\ X\in\mathbb{R}^n,\ A\in\mathfrak{so}(n)\right\}.$$

We identify the above matrix with the triple (a,A,X). It is natural to single out the subalgebras $\mathbb{R}$, $\mathfrak{so}(n)$, and $\mathbb{R}^n$ of $\mathfrak{so}(1,n+1)_{\mathbb{R}p}$. Clearly, $\mathbb{R}$ commutes with $\mathfrak{so}(n)$, and $\mathbb{R}^n$ is an ideal, and also

$$[(a,A,0),(0,0,X)]=(0,0,aX+AX).$$

We get the decomposition[1]

$$\mathfrak{so}(1,n+1)_{\mathbb{R}p}=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n.$$

Each weakly irreducible not irreducible subalgebra $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ is conjugate to some weakly irreducible subalgebra of $\mathfrak{so}(1,n+1)_{\mathbb{R}p}$.

Let $\mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$ be the connected Lie subgroup of $\mathrm{SO}(1,n+1)$ preserving the isotropic line $\mathbb{R}p$. The subalgebras $\mathbb{R}$, $\mathfrak{so}(n)$, and $\mathbb{R}^n\subset\mathfrak{so}(1,n+1)_{\mathbb{R}p}$ correspond to the following connected Lie subgroups:

$$\left\{\left.\begin{pmatrix}a & 0 & 0\\ 0 & \mathrm{id} & 0\\ 0 & 0 & 1/a\end{pmatrix}\,\right|\, a\in\mathbb{R},\ a>0\right\},\qquad \left\{\left.\begin{pmatrix}1 & 0 & 0\\ 0 & f & 0\\ 0 & 0 & 1\end{pmatrix}\,\right|\, f\in\mathrm{SO}(n)\right\},$$

$$\left\{\left.\begin{pmatrix}1 & X^t & -X^tX/2\\ 0 & \mathrm{id} & -X\\ 0 & 0 & 1\end{pmatrix}\,\right|\, X\in\mathbb{R}^n\right\}\subset\mathrm{SO}^0(1,n+1)_{\mathbb{R}p}.$$

[1] Let $\mathfrak{h}$ be a Lie algebra. We write $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ if $\mathfrak{h}$ is the direct sum of the ideals $\mathfrak{h}_1,\mathfrak{h}_2\subset\mathfrak{h}$. We write $\mathfrak{h}=\mathfrak{h}_1\ltimes\mathfrak{h}_2$ if $\mathfrak{h}$ is the direct sum of a subalgebra $\mathfrak{h}_1\subset\mathfrak{h}$ and an ideal $\mathfrak{h}_2\subset\mathfrak{h}$. In the corresponding situations for Lie groups we use the symbols $\times$ and $\leftthreetimes$.

We obtain the decomposition

$$\mathrm{SO}^0(1,n+1)_{\mathbb{R}p} = (\mathbb{R}^+ \times \mathrm{SO}(n)) \leftthreetimes \mathbb{R}^n.$$

Recall that each subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is compact, and one has the decomposition

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h}),$$

where $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ is the commutant of $\mathfrak{h}$, and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$ [118].

The next result is due to Bérard-Bergery and Ikemakhen [21].

Theorem 10. *A subalgebra $\mathfrak{g} \subset \mathfrak{so}(1,n+1)_{\mathbb{R}p}$ is weakly irreducible if and only if $\mathfrak{g}$ is a Lie algebra of one of the following types.*

Type 1:

$$\mathfrak{g}^{1,\mathfrak{h}} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ X \in \mathbb{R}^n,\ A \in \mathfrak{h} \right\},$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is a subalgebra.

Type 2:

$$\mathfrak{g}^{2,\mathfrak{h}} = \mathfrak{h} \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} 0 & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, X \in \mathbb{R}^n,\ A \in \mathfrak{h} \right\},$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is a subalgebra.

Type 3:

$$\begin{aligned} \mathfrak{g}^{3,\mathfrak{h},\varphi} &= \{(\varphi(A), A, 0) \mid A \in \mathfrak{h}\} \ltimes \mathbb{R}^n \\ &= \left\{ \begin{pmatrix} \varphi(A) & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -\varphi(A) \end{pmatrix} \,\middle|\, X \in \mathbb{R}^n,\ A \in \mathfrak{h} \right\}, \end{aligned}$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is a subalgebra such that $\mathfrak{z}(\mathfrak{h}) \neq \{0\}$, and $\varphi\colon \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map such that $\varphi\big|_{\mathfrak{h}'} = 0$.

Type 4:

$$\begin{aligned} \mathfrak{g}^{4,\mathfrak{h},m,\psi} &= \{(0, A, X + \psi(A)) \mid A \in \mathfrak{h},\ X \in \mathbb{R}^m\} \\ &= \left\{ \begin{pmatrix} 0 & X^t & \psi(A)^t & 0 \\ 0 & A & 0 & -X \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, X \in \mathbb{R}^m,\ A \in \mathfrak{h} \right\}, \end{aligned}$$

where there is an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ such that $\mathfrak{h} \subset \mathfrak{so}(m)$, $\dim \mathfrak{z}(\mathfrak{h}) \geqslant n-m$, and $\psi\colon \mathfrak{h} \to \mathbb{R}^{n-m}$ is a surjective linear map with the property that $\psi\big|_{\mathfrak{h}'} = 0$.

The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ associated above with a weakly irreducible subalgebra $\mathfrak{g} \subset \mathfrak{so}(1,n+1)_{\mathbb{R}p}$ is called *the orthogonal part* of the Lie algebra $\mathfrak{g}$.

The proof of this theorem given in [21] is algebraic and does not give any interpretation of the algebras obtained. We give a geometric proof of this result together with an illustrative interpretation.

Theorem 11. *There is a Lie group isomorphism*

$$\mathrm{SO}^0(1,n+1)_{\mathbb{R}p} \simeq \mathrm{Sim}^0(n),$$

where $\mathrm{Sim}^0(n)$ *is the connected Lie group of similarity transformations of* $\mathbb{R}^n$*. Under this isomorphism weakly irreducible Lie subgroups of* $\mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$ *correspond to transitive Lie subgroups of* $\mathrm{Sim}^0(n)$*.*

Proof. We consider the boundary ∂L^{n+1} of the Lobachevskian space as the set of lines of the isotropic cone

$$C = \{X \in \mathbb{R}^{1,n+1} \mid g(X,X) = 0\},$$

that is,

$$\partial L^{n+1} = \{\mathbb{R}X \mid X \in \mathbb{R}^{1,n+1},\ g(X,X) = 0,\ X \neq 0\}.$$

Let us identify ∂L^{n+1} with the n-dimensional unit sphere S^n in the following way. Consider the basis $e_0, e_1, \dots, e_n, e_{n+1}$ of the space $\mathbb{R}^{1,n+1}$, where

$$e_0 = \frac{\sqrt{2}}{2}(p-q), \quad e_{n+1} = \frac{\sqrt{2}}{2}(p+q).$$

We take the vector subspace $E_1 = E \oplus \mathbb{R}e_{n+1} \subset \mathbb{R}^{1,n+1}$. Each isotropic line intersects the affine subspace $e_0 + E_1$ at a unique point. The intersection $(e_0+E_1) \cap C$ constitutes the set

$$\{X \in \mathbb{R}^{1,n+1} \mid x_0 = 1,\ x_1^2 + \dots + x_{n+1}^2 = 1\},$$

which is the n-dimensional sphere S^n. This gives us the identification $\partial L^{n+1} \simeq S^n$.

The group $\mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$ acts on ∂L^{n+1} (as the group of conformal transformations) and it preserves the point $\mathbb{R}p \in \partial L^{n+1}$, that is, $\mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$ acts in the Euclidean space $\mathbb{R}^n \simeq \partial L^{n+1} \setminus \{\mathbb{R}p\}$ as the group of similarity transformations. Indeed, computations show that the elements

$$\begin{pmatrix} a & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 1/a \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & X^t & -X^tX/2 \\ 0 & \mathrm{id} & -X \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$$

act in $\mathbb{R}^n$ as the homothety $Y \mapsto aY$, the special orthogonal transformation $f \in \mathrm{SO}(n)$, and the translation $Y \mapsto Y+X$, respectively. Such transformations generate the Lie group $\mathrm{Sim}^0(n)$. This gives an isomorphism $\mathrm{SO}^0(1,n+1)_{\mathbb{R}p} \simeq \mathrm{Sim}^0(n)$. Next, it is easy to show that a subgroup $G \subset \mathrm{SO}^0(1,n+1)_{\mathbb{R}p}$ does not preserve any proper non-degenerate subspace of $\mathbb{R}^{1,n+1}$ if and only if the corresponding subgroup $G \subset \mathrm{Sim}^0(n)$ does not preserve any proper affine subspace of $\mathbb{R}^n$. The last condition is equivalent to the transitivity of the action of G in $\mathbb{R}^n$ [3], [7]. □

It remains to classify connected transitive Lie subgroups of $\mathrm{Sim}^0(n)$. This is easy to do using results in [3], [7] (see [57]).

Theorem 12. *A connected subgroup* $G \subset \mathrm{Sim}^0(n)$ *is transitive if and only if* G *is conjugate to a group of one of the following types.*

Type 1: $G = (\mathbb{R}^+ \times H) \ltimes \mathbb{R}^n$, *where* $H \subset \mathrm{SO}(n)$ *is a connected Lie subgroup.*
Type 2: $G = H \ltimes \mathbb{R}^n$.
Type 3: $G = (\mathbb{R}^\Phi \times H) \ltimes \mathbb{R}^n$, *where* $\Phi\colon \mathbb{R}^+ \to \mathrm{SO}(n)$ *is a homomorphism and*

$$\mathbb{R}^\Phi = \{a \cdot \Phi(a) \mid a \in \mathbb{R}^+\} \subset \mathbb{R}^+ \times \mathrm{SO}(n)$$

is a group of screw homotheties of $\mathbb{R}^n$.

Type 4: $G = (H \times U^\Psi) \ltimes W$, *where there exists an orthogonal decomposition* $\mathbb{R}^n = U \oplus W$, $H \subset \mathrm{SO}(W)$, $\Psi\colon U \to \mathrm{SO}(W)$ *is an injective homomorphism, and*

$$U^\Psi = \{\Psi(u) \cdot u \mid u \in U\} \subset \mathrm{SO}(W) \times U$$

is a group of screw isometries of $\mathbb{R}^n$.

It is easy to show that the subalgebras $\mathfrak{g} \subset \mathfrak{so}(1, n+1)_{\mathbb{R}p}$ corresponding to the subgroups $G \subset \mathrm{Sim}^0(n)$ in the last theorem exhaust the Lie algebras in Theorem 10. In what follows we will denote the Lie algebra $\mathfrak{so}(1, n+1)_{\mathbb{R}p}$ by $\mathfrak{sim}(n)$.

4. Curvature tensors and classification of Berger algebras

In this section we consider the structure of the spaces $\mathscr{R}(\mathfrak{g})$ of curvature tensors for subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$. Together with the result of Leistner [100] on the classification of weak Berger algebras, this will give a classification of Berger subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$. We then find the curvature tensor of Walker manifolds, that is, manifolds with holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$. The results of this section were published in [55] and [66].

4.1. Algebraic curvature tensors and classification of Berger algebras. In the investigation of the space $\mathscr{R}(\mathfrak{g})$ for subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ there arises the space

$$\begin{aligned}\mathscr{P}(\mathfrak{h}) = \{P \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{h}) \mid g(P(X)Y, Z) + g(P(Y)Z, X)\\ + g(P(Z)X, Y) = 0,\ X, Y, Z \in \mathbb{R}^n\},\end{aligned} \tag{4.1}$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is a subalgebra. The space $\mathscr{P}(\mathfrak{h})$ is called the space of weak curvature tensors for $\mathfrak{h}$. Denote by $L(\mathscr{P}(\mathfrak{h}))$ the vector subspace of $\mathfrak{h}$ spanned by the elements of the form $P(X)$ for all $P \in \mathscr{P}(\mathfrak{h})$ and $X \in \mathbb{R}^n$. It is easy to show [55], [100] that if $R \in \mathscr{R}(\mathfrak{h})$, then $P(\,\cdot\,) = R(\,\cdot\,, Z) \in \mathscr{P}(\mathfrak{h})$ for each $Z \in \mathbb{R}^n$. For this reason the algebra $\mathfrak{h}$ is called a weak Berger algebra if $L(\mathscr{P}(\mathfrak{h})) = \mathfrak{h}$. An $\mathfrak{h}$-module structure is introduced in the space $\mathscr{P}(\mathfrak{h})$ in the natural way:

$$P_\xi(X) = [\xi, P(X)] - P(\xi X),$$

where $P \in \mathscr{P}(\mathfrak{h})$, $\xi \in \mathfrak{h}$, and $X \in \mathbb{R}^n$. This implies that the subspace $L(\mathscr{P}(\mathfrak{h})) \subset \mathfrak{h}$ is an ideal in $\mathfrak{h}$.

It is convenient to identify the Lie algebra $\mathfrak{so}(1, n+1)$ with the space $\wedge^2\mathbb{R}^{1,n+1}$ of bivectors in such a way that

$$(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X, \qquad X, Y, Z \in \mathbb{R}^{1,n+1}.$$

Then the element $(a, A, X) \in \mathfrak{sim}(n)$ corresponds to the bivector $-ap\wedge q+A-p\wedge X$, where $A \in \mathfrak{so}(n) \simeq \wedge^2\mathbb{R}^n$.

The next theorem, from [55], provides the structure of the space of curvature tensors for weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$.

Theorem 13. *Each curvature tensor $R \in \mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}})$ is uniquely determined by the elements*

$$\lambda \in \mathbb{R}, \quad \vec{v} \in \mathbb{R}^n, \quad R_0 \in \mathscr{R}(\mathfrak{h}), \quad P \in \mathscr{P}(\mathfrak{h}), \quad T \in \odot^2\mathbb{R}^n$$

as follows:

$$R(p,q) = -\lambda p\wedge q - p\wedge \vec{v}, \quad R(X,Y) = R_0(X,Y) + p\wedge(P(X)Y - P(Y)X), \tag{4.2}$$

$$R(X,q) = -g(\vec{v},X)p\wedge q + P(X) - p\wedge T(X), \quad R(p,X) = 0, \tag{4.3}$$

$X, Y \in \mathbb{R}^n$. *In particular, there exists an $\mathfrak{h}$-module isomorphism*

$$\mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}}) \simeq \mathbb{R}\oplus\mathbb{R}^n\oplus\odot^2\mathbb{R}^n\oplus\mathscr{R}(\mathfrak{h})\oplus\mathscr{P}(\mathfrak{h}).$$

Further,

$$\begin{aligned}
\mathscr{R}(\mathfrak{g}^{2,\mathfrak{h}}) &= \{R\in\mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}}) \mid \lambda=0,\ \vec{v}=0\},\\
\mathscr{R}(\mathfrak{g}^{3,\mathfrak{h},\varphi}) &= \{R\in\mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}}) \mid \lambda=0,\ R_0\in\mathscr{R}(\ker\varphi),\ g(\vec{v},\cdot\,)=\varphi(P(\,\cdot\,))\},\\
\mathscr{R}(\mathfrak{g}^{4,\mathfrak{h},m,\psi}) &= \{R\in\mathscr{R}(\mathfrak{g}^{2,\mathfrak{h}}) \mid R_0\in\mathscr{R}(\ker\psi),\ \mathrm{pr}_{\mathbb{R}^{n-m}}\circ T=\psi\circ P\}.
\end{aligned}$$

Corollary 1 [55]. *A weakly irreducible subalgebra $\mathfrak{g}\subset\mathfrak{sim}(n)$ is a Berger algebra if and only if its orthogonal part $\mathfrak{h}\subset\mathfrak{so}(n)$ is a weak Berger algebra.*

Corollary 2 [55]. *A weakly irreducible subalgebra $\mathfrak{g}\subset\mathfrak{sim}(n)$ whose orthogonal part $\mathfrak{h}\subset\mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold is a Berger algebra.*

Corollary 1 reduces the classification problem for Berger algebras of Lorentzian manifolds to the classification problem for weak Berger algebras.

Theorem 14 [55]. (I) *For each weak Berger algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ there exist an orthogonal decomposition*

$$\mathbb{R}^n = \mathbb{R}^{n_1}\oplus\cdots\oplus\mathbb{R}^{n_s}\oplus\mathbb{R}^{n_{s+1}} \tag{4.4}$$

and a corresponding decomposition of $\mathfrak{h}$ into a direct sum of ideals

$$\mathfrak{h} = \mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_s\oplus\{0\} \tag{4.5}$$

such that $\mathfrak{h}_i(\mathbb{R}^{n_j})=0$ for $i\neq j$, $\mathfrak{h}_i\subset\mathfrak{so}(n_i)$, and the representation of $\mathfrak{h}_i$ on $\mathbb{R}^{n_i}$ is irreducible.

(II) *Suppose that $\mathfrak{h}\subset\mathfrak{so}(n)$ is a subalgebra with the decomposition in part* (I). *Then*

$$\mathscr{P}(\mathfrak{h}) = \mathscr{P}(\mathfrak{h}_1)\oplus\cdots\oplus\mathscr{P}(\mathfrak{h}_s).$$

Bérard-Bergery and Ikemakhen [21] proved that the orthogonal part $\mathfrak{h} \subset \mathfrak{so}(n)$ of a holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$ admits the decomposition in part (I) of the theorem.

Corollary 3 [55]. *Suppose that* $\mathfrak{h} \subset \mathfrak{so}(n)$ *is a subalgebra admitting a decomposition as in part* (I) *of Theorem 14. Then* $\mathfrak{h}$ *is a weak Berger algebra if and only if the algebra* $\mathfrak{h}_i$ *is a weak Berger algebra for all* $i = 1, \dots, s$.

Thus, it is sufficient to consider irreducible weak Berger algebras $\mathfrak{h} \subset \mathfrak{so}(n)$. It turns out that these algebras are irreducible holonomy algebras of Riemannian manifolds. This far from non-trivial statement was proved by Leistner [100].

Theorem 15 [100]. *An irreducible subalgebra* $\mathfrak{h} \subset \mathfrak{so}(n)$ *is a weak Berger algebra if and only if it is the holonomy algebra of a Riemannian manifold.*

We will discuss the proof of this theorem below in § 6. From Corollary 1 and Theorem 15 we get a classification of weakly irreducible not irreducible Berger algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$.

Theorem 16. *A subalgebra* $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ *is a weakly irreducible not irreducible Berger algebra if and only if* $\mathfrak{g}$ *is conjugate to one of the subalgebras* $\mathfrak{g}^{1,\mathfrak{h}}$, $\mathfrak{g}^{2,\mathfrak{h}}$, $\mathfrak{g}^{3,\mathfrak{h},\varphi}$, $\mathfrak{g}^{4,\mathfrak{h},m,\psi} \subset \mathfrak{sim}(n)$, *where* $\mathfrak{h} \subset \mathfrak{so}(n)$ *is the holonomy algebra of a Riemannian manifold.*

Let us return to the statement of Theorem 13. Note that the elements determining $R \in \mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}})$ in Theorem 13 depend on the choice of the vectors $p, q \in \mathbb{R}^{1,n+1}$. Consider a real number $\mu \neq 0$, the vector $p' = \mu p$, and an arbitrary isotropic vector q' such that $g(p', q') = 1$. There exists a unique vector $W \in E$ such that

$$q' = \frac{1}{\mu}\left(-\frac{1}{2} g(W, W)p + W + q\right).$$

The corresponding space E' has the form

$$E' = \{-g(X, W)p + X \mid X \in E\}.$$

We will consider the map

$$E \ni X \mapsto X' = -g(X, W)p + X \in E'.$$

It is easy to show that the tensor R is determined by the elements $\widetilde{\lambda}$, $\widetilde{v}$, $\widetilde{R}_0$, $\widetilde{P}$, $\widetilde{T}$, where, for example, we have

$$\begin{gathered} \widetilde{\lambda} = \lambda, \quad \widetilde{v} = \frac{1}{\mu}(\vec{v} - \lambda W)', \quad \widetilde{P}(X') = \frac{1}{\mu}(P(X) + R_0(X, W))', \\ \widetilde{R}_0(X', Y')Z' = (R_0(X, Y)Z)'. \end{gathered} \tag{4.6}$$

Let $R \in \mathscr{R}(\mathfrak{g}^{1,\mathfrak{h}})$. The corresponding Ricci tensor has the form

$$\operatorname{Ric}(p, q) = \lambda, \qquad \operatorname{Ric}(X, Y) = \operatorname{Ric}(R_0)(X, Y), \tag{4.7}$$

$$\operatorname{Ric}(X, q) = g(X, \vec{v} - \widetilde{\operatorname{Ric}}(P)), \qquad \operatorname{Ric}(q, q) = -\operatorname{tr} T, \tag{4.8}$$

where $\widetilde{\mathrm{Ric}}(P)=\sum_{i=1}^n P(e_i)e_i$. The scalar curvature satisfies

$$s=2\lambda+s_0,$$

where s_0 is the scalar curvature of the tensor R_0.

The Ricci operator has the form

$$\mathrm{Ric}(p)=\lambda p,\qquad \mathrm{Ric}(X)=g(X,\vec{v}-\widetilde{\mathrm{Ric}}(P))p+\mathrm{Ric}(R_0)(X), \tag{4.9}$$

$$\mathrm{Ric}(q)=-(\mathrm{tr}\,T)p-\widetilde{\mathrm{Ric}}(P)+\vec{v}+\lambda q. \tag{4.10}$$

4.2. Curvature tensor of Walker manifolds. Each Lorentzian manifold (M,g) with holonomy algebra $\mathfrak{g}\subset\mathfrak{sim}(n)$ (locally) admits a parallel distribution of isotropic lines ℓ. These manifolds are called Walker manifolds [37], [120].

The vector bundle $\mathscr{E}=\ell^\perp/\ell$ is called the screen bundle. The holonomy algebra of the induced connection in $\mathscr{E}$ coincides with the orthogonal part $\mathfrak{h}\subset\mathfrak{so}(n)$ of the holonomy algebra of the manifold (M,g).

On a Walker manifold (M,g) there exist local coordinates $v,x^1,\dots,x^n,u$ such that the metric g is of the from

$$g=2\,dv\,du+h+2A\,du+H\,(du)^2, \tag{4.11}$$

where $h=h_{ij}(x^1,\dots,x^n,u)\,dx^i\,dx^j$ is a family of Riemannian metrics depending on the parameter u, $A=A_i(x^1,\dots,x^n,u)\,dx^i$ is a family of 1-forms depending on u, and H is a local function on M.

Note that the holonomy algebra of the metric h is contained in the orthogonal part $\mathfrak{h}\subset\mathfrak{so}(n)$ of the holonomy algebra of the metric g, and this inclusion can be strict.

The vector field ∂_v defines a parallel distribution of isotropic lines and is recurrent, that is,

$$\nabla\partial_v=\frac{1}{2}\partial_v H\,du\otimes\partial_v.$$

Therefore, ∂_v is proportional to a parallel vector field if and only if $d(\partial_v H\,du)=0$, which is equivalent to the equalities

$$\partial_v\partial_i H=\partial_v^2 H=0.$$

In this case the coordinates can be chosen in such a way that $\nabla\partial_v=0$ and $\partial_v H=0$. The holonomy algebras of types 2 and 4 annihilate the vector p, and consequently the corresponding manifolds admit (local) parallel isotropic vector fields, and the local coordinates can be chosen in such a way that $\partial_v H=0$. In contrast, the holonomy algebras of types 1 and 3 do not annihilate this vector, and consequently the corresponding manifolds admit only recurrent isotropic vector fields, and in this case $d(\partial_v H\,du)\neq 0$.

An important class of Walker manifolds is formed by pp-waves (gravitational plane waves with parallel propagation), which are defined locally by (4.11) with $A=0$, $h=\sum_{i=1}^n(dx^i)^2$, and $\partial_v H=0$. The pp-waves are precisely the Walker manifolds with commutative holonomy algebras $\mathfrak{g}\subset\mathbb{R}^n\subset\mathfrak{sim}(n)$.

Boubel [32] constructed the coordinates

$$v, x_1 = (x_1^1, \dots, x_1^{n_1}), \dots, x_{s+1} = (x_{s+1}^1, \dots, x_{s+1}^{n_{s+1}}), u, \tag{4.12}$$

corresponding to the decomposition (4.4). This means that

$$h = h_1 + \cdots + h_{s+1}, \quad h_\alpha = \sum_{i,j=1}^{n_\alpha} h_{\alpha ij}\, dx_\alpha^i\, dx_\alpha^j, \quad h_{s+1} = \sum_{i=1}^{n_{s+1}} (dx_{s+1}^i)^2, \tag{4.13}$$

$$A = \sum_{\alpha=1}^{s+1} A_\alpha, \quad A_\alpha = \sum_{k=1}^{n_\alpha} A_k^\alpha\, dx_\alpha^k, \quad A_{s+1} = 0,$$

$$\frac{\partial}{\partial x_\beta^k} h_{\alpha ij} = \frac{\partial}{\partial x_\beta^k} A_i^\alpha = 0 \quad \text{if } \beta \neq \alpha. \tag{4.14}$$

We consider the field of frames

$$p = \partial_v, \quad X_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2} H \partial_v \tag{4.15}$$

and the distribution E generated by the vector fields $X_1, \cdots, X_n$. The fibres of this distribution can be identified with the tangent spaces to the Riemannian manifolds with the Riemannian metrics $h(u)$. Denote by R_0 the tensor corresponding to the family of the curvature tensors of the metrics $h(u)$ under this identification. Similarly, denote by $\mathrm{Ric}(h)$ the corresponding Ricci endomorphism acting on sections of E. Now the curvature tensor R of the metric g is uniquely determined by a function λ, a section $\vec{v} \in \Gamma(E)$, a symmetric field of endomorphisms $T \in \Gamma(\mathrm{End}(E))$ with $T^* = T$, the curvature tensor $R_0 = R(h)$, and a tensor $P \in \Gamma(E^* \otimes \mathfrak{so}(E))$. These tensors can be expressed in terms of the coefficients of the metric (4.11). Let $P(X_k)X_j = P_{jk}^i X_i$ and $T(X_j) = \sum_i T_{ij} X_j$. Then

$$h_{il} P_{jk}^l = g(R(X_k, q)X_j, X_i), \quad T_{ij} = -g(R(X_i, q)q, X_j).$$

Direct computations show that

$$\lambda = \frac{1}{2} \partial_v^2 H, \quad \vec{v} = \frac{1}{2} (\partial_i \partial_v H - A_i \partial_v^2 H) h^{ij} X_j, \tag{4.16}$$

$$h_{il} P_{jk}^l = -\frac{1}{2} \nabla_k F_{ij} + \frac{1}{2} \nabla_k \dot{h}_{ij} - \dot{\Gamma}_{kj}^l h_{li}, \tag{4.17}$$

$$\begin{aligned} T_{ij} = {} & \frac{1}{2} \nabla_i \nabla_j H - \frac{1}{4} (F_{ik} + \dot{h}_{ik})(F_{jl} + \dot{h}_{jl}) h^{kl} - \frac{1}{4} (\partial_v H)(\nabla_i A_j + \nabla_j A_i) \\ & - \frac{1}{2} (A_i \partial_j \partial_v H + A_j \partial_i \partial_v H) - \frac{1}{2} (\nabla_i \dot{A}_j + \nabla_j \dot{A}_i) \\ & + \frac{1}{2} A_i A_j \partial_v^2 H + \frac{1}{2} \ddot{h}_{ij} + \frac{1}{4} \dot{h}_{ij} \partial_v H, \end{aligned} \tag{4.18}$$

where

$$F = dA, \quad F_{ij} = \partial_i A_j - \partial_j A_i,$$

is the differential of the 1-form A, the covariant derivatives are taken with respect to the metric h, and the dot denotes the partial derivative with respect to the variable u. In the case of h, A, and H independent of u the curvature tensor of the metric (4.11) was found in [76]. Also in [76], the Ricci tensor was found for an arbitrary metric (4.11).

It is important to note that the Walker coordinates are not defined canonically; for instance, a useful observation in [76] shows that if

$$H = \lambda v^2 + vH_1 + H_0, \qquad \lambda \in \mathbb{R}, \quad \partial_v H_1 = \partial_v H_0 = 0,$$

then the coordinate transformation

$$v \mapsto v - f(x^1, \dots, x^n, u), \quad x^i \mapsto x^i, \quad u \mapsto u$$

changes the metric (4.11) in the following way:

$$A_i \mapsto A_i + \partial_i f, \quad H_1 \mapsto H_1 + 2\lambda f, \quad H_0 \mapsto H_0 + H_1 f + \lambda f^2 + 2\dot{f}. \tag{4.19}$$

5. The spaces of weak curvature tensors

Although Leistner proved that the subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ spanned by the images of elements of the space $\mathscr{P}(\mathfrak{h})$ are exhausted by the holonomy algebras of Riemannian spaces, he did not find the spaces $\mathscr{P}(\mathfrak{h})$ themselves. Here we present the results of computations of these spaces in [59], giving the complete structure of the spaces of curvature tensors for the holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$.

Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible subalgebra. We consider the $\mathfrak{h}$-equivariant map

$$\widetilde{\mathrm{Ric}}\colon \mathscr{P}(\mathfrak{h}) \to \mathbb{R}^n, \qquad \widetilde{\mathrm{Ric}}(P) = \sum_{i=1}^{n} P(e_i)e_i.$$

The definition of this map does not depend on the choice of the orthogonal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. Denote by $\mathscr{P}_0(\mathfrak{h})$ the kernel of the map $\widetilde{\mathrm{Ric}}$, and let $\mathscr{P}_1(\mathfrak{h})$ be the orthogonal complement of this space in $\mathscr{P}(\mathfrak{h})$. Thus,

$$\mathscr{P}(\mathfrak{h}) = \mathscr{P}_0(\mathfrak{h}) \oplus \mathscr{P}_1(\mathfrak{h}).$$

Since the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible and the map $\widetilde{\mathrm{Ric}}$ is $\mathfrak{h}$-equivariant, the space $\mathscr{P}_1(\mathfrak{h})$ is either trivial or isomorphic to $\mathbb{R}^n$. The spaces $\mathscr{P}(\mathfrak{h})$ for $\mathfrak{h} \subset \mathfrak{u}(n/2)$ are found in [100]. In [59] we compute the spaces $\mathscr{P}(\mathfrak{h})$ for the remaining holonomy algebras of Riemannian manifolds. The main result is Table 1, where the spaces $\mathscr{P}(\mathfrak{h})$ are given for all irreducible holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(n)$ of Riemannian manifolds (for a compact Lie algebra $\mathfrak{h}$ the symbol V_Λ denotes the irreducible representation of $\mathfrak{h}$ given by the irreducible representation of the Lie algebra $\mathfrak{h} \otimes \mathbb{C}$ with highest weight Λ, and $(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$ denotes the subspace of $\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m$ consisting of the tensors such that contraction of the upper index with any lower index gives zero).

Consider the natural $\mathfrak{h}$-equivariant map

$$\tau\colon \mathbb{R}^n \otimes \mathscr{R}(\mathfrak{h}) \to \mathscr{P}(\mathfrak{h}), \qquad \tau(u \otimes R) = R(\,\cdot\,, u).$$

The next theorem will be used to find the explicit form of certain $P \in \mathscr{P}(\mathfrak{h})$. The proof of the theorem follows from results in [2], [100], and Table 1.

Table 1. The spaces $\mathscr{P}(\mathfrak{h})$ for irreducible holonomy algebras of Riemannian manifolds $\mathfrak{h} \subset \mathfrak{so}(n)$

$\mathfrak{h} \subset \mathfrak{so}(n)$	$\mathscr{P}_1(\mathfrak{h})$	$\mathscr{P}_0(\mathfrak{h})$	$\dim \mathscr{P}_0(\mathfrak{h})$
$\mathfrak{so}(2)$	$\mathbb{R}^2$	0	0
$\mathfrak{so}(3)$	$\mathbb{R}^3$	$V_{4\pi_1}$	5
$\mathfrak{so}(4)$	$\mathbb{R}^4$	$V_{3\pi_1+\pi_1'} \oplus V_{\pi_1+3\pi_1'}$	16
$\mathfrak{so}(n)$, $n \geqslant 5$	$\mathbb{R}^n$	$V_{\pi_1+\pi_2}$	$\frac{(n-2)n(n+2)}{3}$
$\mathfrak{u}(m)$, $n = 2m \geqslant 4$	$\mathbb{R}^n$	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{su}(m)$, $n = 2m \geqslant 4$	0	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, $n = 4m \geqslant 8$	$\mathbb{R}^n$	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$\mathfrak{sp}(m)$, $n = 4m \geqslant 8$	0	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$G_2 \subset \mathfrak{so}(7)$	0	$V_{\pi_1+\pi_2}$	64
$\mathfrak{spin}(7) \subset \mathfrak{so}(8)$	0	$V_{\pi_2+\pi_3}$	112
$\mathfrak{h} \subset \mathfrak{so}(n)$, $n \geqslant 4$, is a symmetric Berger algebra	$\mathbb{R}^n$	0	0

Theorem 17. *For an arbitrary irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$, $n \geqslant 4$, the $\mathfrak{h}$-equivariant map $\tau\colon \mathbb{R}^n \otimes \mathscr{R}(\mathfrak{h}) \to \mathscr{P}(\mathfrak{h})$ is surjective. Moreover, $\tau(\mathbb{R}^n \otimes \mathscr{R}_0(\mathfrak{h})) = \mathscr{P}_0(\mathfrak{h})$ and $\tau(\mathbb{R}^n \otimes \mathscr{R}_1(\mathfrak{h})) = \mathscr{P}_1(\mathfrak{h})$.*

Let $n \geqslant 4$, and let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible subalgebra. From Theorem 17 it follows that an arbitrary $P \in \mathscr{P}_1(\mathfrak{h})$ can be written in the form $R(\,\cdot\,, x)$, where $R \in \mathscr{R}_0(\mathfrak{h})$ and $x \in \mathbb{R}^n$. Similarly, any $P \in \mathscr{P}_0(\mathfrak{h})$ can be represented in the form $\sum_i R_i(\,\cdot\,, x_i)$ for some $R_i \in \mathscr{R}_1(\mathfrak{h})$ and $x_i \in \mathbb{R}^n$.

The explicit form of certain $P \in \mathscr{P}(\mathfrak{h})$. Using the results obtained above and results from [2], we can now find explicitly the spaces $\mathscr{P}(\mathfrak{h})$.

From the results in [100] it follows that

$$\mathscr{P}(\mathfrak{u}(m)) \simeq \odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m.$$

We give an explicit description of this isomorphism. Let

$$S \in \odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m \subset (\mathbb{C}^m)^* \otimes \mathfrak{gl}(m, \mathbb{C}).$$

We consider the identification

$$\mathbb{C}^m = \mathbb{R}^{2m} = \mathbb{R}^m \oplus i\mathbb{R}^m$$

and chose a basis $e_1, \dots, e_m$ of the space $\mathbb{R}^m$. Define complex numbers S_{abc}, $a, b, c = 1, \dots, m$, such that

$$S(e_a)e_b = \sum_c S_{acb} e_c.$$

We have $S_{abc} = S_{cba}$. Let $S_1\colon \mathbb{R}^{2m} \to \mathfrak{gl}(2m,\mathbb{R})$ be the map defined by the conditions

$$S_1(e_a)e_b = \sum_c \overline{S_{abc}} e_c, \qquad S_1(ie_a) = -iS_1(e_a), \qquad S_1(e_a)ie_b = iS_1(e_a)e_b.$$

It is easy to check that

$$P = S - S_1\colon \mathbb{R}^{2m} \to \mathfrak{gl}(2m,\mathbb{R})$$

belongs to $\mathscr{P}(\mathfrak{u}(n))$, and each element of the space $\mathscr{P}(\mathfrak{u}(n))$ is of this form. The element obtained belongs to the space $\mathscr{P}(\mathfrak{su}(n))$ if and only if $\sum_b S_{abb} = 0$ for all $a = 1,\dots,m$, that is, $S \in (\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$. If $m = 2k$, that is, $n = 4k$, then P belongs to $\mathscr{P}(\mathfrak{sp}(k))$ if and only if $S(e_a) \in \mathfrak{sp}(2k,\mathbb{C})$, $a = 1,\dots,m$, that is,

$$S \in (\mathfrak{sp}(2k,\mathbb{C}))^{(1)} \simeq \odot^3(\mathbb{C}^{2k})^*.$$

In [72] it is shown that each $P \in \mathscr{P}(\mathfrak{u}(m))$ satisfies

$$g(\widetilde{\mathrm{Ric}}(P), X) = -\operatorname{tr}_{\mathbb{C}} P(JX), \qquad X \in \mathbb{R}^{2m}.$$

And in [2] it is shown that an arbitrary $R \in \mathscr{R}_1(\mathfrak{so}(n)) \oplus \mathscr{R}'(\mathfrak{so}(n))$ has the form $R = R_S$, where $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear map, and

$$R_S(X,Y) = SX \wedge Y + X \wedge SY. \tag{5.1}$$

It is easy to check that

$$\tau\big(\mathbb{R}^n, \mathscr{R}_1(\mathfrak{so}(n)) \oplus \mathscr{R}'(\mathfrak{so}(n))\big) = \mathscr{P}(\mathfrak{so}(n)).$$

This equality and (5.1) show that the space $\mathscr{P}(\mathfrak{so}(n))$ is spanned by the elements P of the form

$$P(y) = Sy \wedge x + y \wedge Sx,$$

where $x \in \mathbb{R}^n$ and $S \in \odot^2\mathbb{R}^n$ are fixed, and $y \in \mathbb{R}^n$ is an arbitrary vector. For such P we have $\widetilde{\mathrm{Ric}}(P) = (\operatorname{tr} S - S)x$. This means that the space $\mathscr{P}_0(\mathfrak{so}(n))$ is spanned by elements P of the form

$$P(y) = Sy \wedge x,$$

where $x \in \mathbb{R}^n$ and $S \in \odot^2\mathbb{R}^n$ satisfy $\operatorname{tr} S = 0$ and $Sx = 0$, and $y \in \mathbb{R}^n$ is an arbitrary vector.

The isomorphism $\mathscr{P}_1(\mathfrak{so}(n)) \simeq \mathbb{R}^n$ is defined as follows: $x \in \mathbb{R}^n$ corresponds to the element $P = x \wedge \cdot \in \mathscr{P}_1(\mathfrak{so}(n))$, that is, $P(y) = x \wedge y$ for all $y \in \mathbb{R}^n$.

Each $P \in \mathscr{P}_1(\mathfrak{u}(m))$ has the form

$$P(y) = -\frac{1}{2}g(Jx,y)J + \frac{1}{4}(x \wedge y + Jx \wedge Jy),$$

where J is a complex structure on $\mathbb{R}^{2m}$, the vector $x \in \mathbb{R}^{2m}$ is fixed, and the vector $y \in \mathbb{R}^{2m}$ is arbitrary.

Each $P \in \mathscr{P}_1(\mathfrak{sp}(m)\oplus\mathfrak{sp}(1))$ has the form

$$P(y) = -\frac{1}{2}\sum_{\alpha=1}^{3} g(J_\alpha x, y)J_\alpha + \frac{1}{4}\biggl(x\wedge y + \sum_{\alpha=1}^{3} J_\alpha x\wedge J_\alpha y\biggr),$$

where (J_1, J_2, J_3) is a quaternionic structure on $\mathbb{R}^{4m}$, $x \in \mathbb{R}^{4m}$ is fixed, and $y \in \mathbb{R}^{4m}$ is an arbitrary vector.

For the adjoint representation $\mathfrak{h} \subset \mathfrak{so}(\mathfrak{h})$ of a simple compact Lie algebra $\mathfrak{h}$ different from $\mathfrak{so}(3)$, an arbitrary element $P \in \mathscr{P}(\mathfrak{h}) = \mathscr{P}_1(\mathfrak{h})$ has the form

$$P(y) = [x, y].$$

If $\mathfrak{h} \subset \mathfrak{so}(n)$ is a symmetric Berger algebra, then

$$\mathscr{P}(\mathfrak{h}) = \mathscr{P}_1(\mathfrak{h}) = \{R(\,\cdot\,, x) \mid x \in \mathbb{R}^n\},$$

where R is a generator of the space $\mathscr{R}(\mathfrak{h}) \simeq \mathbb{R}$.

In the general case let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible subalgebra, and let $P \in \mathscr{P}_1(\mathfrak{h})$. Then $\widetilde{\mathrm{Ric}}(P)\wedge\cdot \in \mathscr{P}_1(\mathfrak{so}(n))$. Moreover, it is easy to check that

$$\widetilde{\mathrm{Ric}}\biggl(P + \frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot\biggr) = 0,$$

that is,

$$P + \frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot \in \mathscr{P}_0(\mathfrak{so}(n)).$$

Thus, the inclusion

$$\mathscr{P}_1(\mathfrak{h}) \subset \mathscr{P}(\mathfrak{so}(n)) = \mathscr{P}_0(\mathfrak{so}(n)) \oplus \mathscr{P}_1(\mathfrak{so}(n))$$

has the form

$$P \in P_1(\mathfrak{h}) \mapsto \biggl(P + \frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot, -\frac{1}{n-1}\widetilde{\mathrm{Ric}}(P)\wedge\cdot\biggr) \in \mathscr{P}_0(\mathfrak{so}(n)) \oplus \mathscr{P}_1(\mathfrak{so}(n)).$$

This construction defines the tensor $W = P + (1/(n-1))\widetilde{\mathrm{Ric}}(P)\wedge\cdot$ analogous to the Weyl tensor for $P \in \mathscr{P}(\mathfrak{h})$, and W is a certain component of the Weyl tensor of a Lorentzian manifold.

6. On the classification of weak Berger algebras

A crucial point in the classification of holonomy algebras of Lorentzian manifolds is Leistner's result on the classification of irreducible weak Berger algebras $\mathfrak{h} \subset \mathfrak{so}(n)$. He classified all such subalgebras, and it turned out that the list obtained coincides with the list of irreducible holonomy algebras of Riemannian manifolds. The natural problem arises of obtaining a simple direct proof of this fact. In [68] we gave such a proof for the case of semisimple not simple Lie algebras $\mathfrak{h} \subset \mathfrak{so}(n)$.

In the paper [55], the first version of which was published in April 2003 on the web cite www.arXiv.org, Leistner's Theorem 15 was proved for $n \leqslant 9$. To this end, irreducible subalgebras $\mathfrak{h}\subset\mathfrak{so}(n)$ with $n\leqslant 9$ were listed (see Table 2). The second column of the table contains the irreducible holonomy algebras of Riemannian manifolds, and the third column contains algebras that are not the holonomy algebras of Riemannian manifolds.

For a semisimple compact Lie algebra $\mathfrak{h}$ we denote by $\pi^{\mathbb{K}}_{\Lambda_1,\dots,\Lambda_l}(\mathfrak{h})$ the image of the representation $\pi^{\mathbb{K}}_{\Lambda_1,\dots,\Lambda_l}\colon\mathfrak{h}\to\mathfrak{so}(n)$ that is uniquely determined by the complex representation $\rho_{\Lambda_1,\dots,\Lambda_l}\colon\mathfrak{h}(\mathbb{C})\to\mathfrak{gl}(U)$ given by the numerical labels $\Lambda_1,\dots,\Lambda_l$ on the Dynkin diagram, where $\mathfrak{h}(\mathbb{C})$ is the complexification of the algebra $\mathfrak{h}$, U is a certain complex vector space, and $\mathbb{K}=\mathbb{R}$, $\mathbb{H}$, or $\mathbb{C}$ if $\rho_{\Lambda_1,\dots,\Lambda_l}$ is real, quaternionic, or complex, respectively. The symbol $\mathfrak{t}$ denotes the 1-dimensional center.

Table 2. Irreducible subalgebras of $\mathfrak{so}(n)$ ($n\leqslant 9$)

n	irreducible holonomy algebras of n-dimensional Riemannian manifolds	other irreducible subalgebras of $\mathfrak{so}(n)$
$n=1$		
$n=2$	$\mathfrak{so}(2)$	
$n=3$	$\pi^{\mathbb{R}}_2(\mathfrak{so}(3))$	
$n=4$	$\pi^{\mathbb{R}}_{1,1}(\mathfrak{so}(3)\oplus\mathfrak{so}(3))$, $\pi^{\mathbb{C}}_1(\mathfrak{su}(2))$, $\pi^{\mathbb{C}}_1(\mathfrak{su}(2))\oplus\mathfrak{t}$	
$n=5$	$\pi^{\mathbb{R}}_{1,0}(\mathfrak{so}(5))$, $\pi^{\mathbb{R}}_4(\mathfrak{so}(3))$	
$n=6$	$\pi^{\mathbb{R}}_{1,0,0}(\mathfrak{so}(6))$, $\pi^{\mathbb{C}}_{1,0}(\mathfrak{su}(3))$, $\pi^{\mathbb{C}}_{1,0}(\mathfrak{su}(3))\oplus\mathfrak{t}$	
$n=7$	$\pi^{\mathbb{R}}_{1,0,0}(\mathfrak{so}(7))$, $\pi^{\mathbb{R}}_{1,0}(\mathfrak{g}_2)$	$\pi^{\mathbb{R}}_6(\mathfrak{so}(3))$
$n=8$	$\pi^{\mathbb{R}}_{1,0,0,0}(\mathfrak{so}(8))$, $\pi^{\mathbb{C}}_{1,0}(\mathfrak{su}(4))$, $\pi^{\mathbb{C}}_{1,0}(\mathfrak{su}(4))\oplus\mathfrak{t}$, $\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2))$, $\pi^{\mathbb{R}}_{1,0,1}(\mathfrak{sp}(2)\oplus\mathfrak{sp}(1))$, $\pi^{\mathbb{R}}_{0,0,1}(\mathfrak{so}(7))$, $\pi^{\mathbb{R}}_{1,3}(\mathfrak{so}(3)\oplus\mathfrak{so}(3))$, $\pi^{\mathbb{R}}_{1,1}(\mathfrak{su}(3))$	$\pi^{\mathbb{C}}_3(\mathfrak{so}(3))$, $\pi^{\mathbb{C}}_3(\mathfrak{so}(3))\oplus\mathfrak{t}$, $\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2))\oplus\mathfrak{t}$
$n=9$	$\pi^{\mathbb{R}}_{1,0,0,0}(\mathfrak{so}(9))$, $\pi^{\mathbb{R}}_{2,2}(\mathfrak{so}(3)\oplus\mathfrak{so}(3))$	$\pi^{\mathbb{R}}_8(\mathfrak{so}(3))$

For algebras that are not the holonomy algebras of Riemannian manifolds a computer program was used to find the spaces $\mathscr{P}(\mathfrak{h})$ as solutions of the corresponding systems of linear equations. It turned out that

$$\mathscr{P}(\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2)))=\mathscr{P}(\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2))\oplus\mathfrak{t}),$$

that is, $L(\mathscr{P}(\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2))\oplus\mathfrak{t}))=\pi^{\mathbb{H}}_{1,0}(\mathfrak{sp}(2))$, and $\mathfrak{sp}(2)\oplus\mathfrak{t}$ is not a weak Berger algebra. For the other algebras in the third column we have $\mathscr{P}(\mathfrak{h})=0$. Hence, the Lie algebras in the third column of Table 2 are not weak Berger algebras.

It turned out that by that time Leistner had already proved Theorem 15 and published its proof as a preprint in the cases when n is even and the representation $\mathfrak{h}\subset\mathfrak{so}(n)$ is of complex type, that is, $\mathfrak{h}\subset\mathfrak{u}(n/2)$. In this case $\mathscr{P}(\mathfrak{h})\simeq(\mathfrak{h}\otimes\mathbb{C})^{(1)}$, where $(\mathfrak{h}\otimes\mathbb{C})^{(1)}$ is the first prolongation of the subalgebra $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{gl}(n/2,\mathbb{C})$. Using this fact and the classification of irreducible representations with non-trivial prolongations, Leistner showed that each weak Berger subalgebra $\mathfrak{h}\subset\mathfrak{u}(n/2)$ is the holonomy algebra of a Riemannian manifold.

The case of subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ of real type (that is, not of complex type) is much more difficult. In this case Leistner considered the complexified representation $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$, which is irreducible. Using the classification of irreducible representations of complex semisimple Lie algebras, he found a criterion, in terms of the weights of these representations, for $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ to be a weak Berger algebra. He then considered simple Lie algebras $\mathfrak{h} \otimes \mathbb{C}$ case by case, and then semisimple Lie algebras (the problem was reduced to the semisimple Lie algebras of the form $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$, where $\mathfrak{k}$ is simple, and then again different possibilities for $\mathfrak{k}$ were considered). The complete proof is published in [100].

We consider the case of semisimple not simple irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ with irreducible complexification $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$. In a simple way we show that it suffices to treat the case when $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$, where $\mathfrak{k} \subsetneq \mathfrak{sp}(2m, \mathbb{C})$ is a proper irreducible subalgebra, and the representation space is the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^{2m}$. We show that in this case $\mathscr{P}(\mathfrak{h})$ coincides with $\mathbb{C}^2 \otimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is the first Tanaka prolongation of the non-positively graded Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0;$$

here $\mathfrak{g}_{-2} = \mathbb{C}$, $\mathfrak{g}_{-1} = \mathbb{C}^{2m}$, $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}\,\mathrm{id}_{\mathbb{C}^{2m}}$, and the grading is defined by the element $-\mathrm{id}_{\mathbb{C}^{2m}}$. We prove that if $\mathscr{P}(\mathfrak{h})$ is non-trivial, then $\mathfrak{g}_1$ is isomorphic to $\mathbb{C}^{2m}$, the second Tanaka prolongation $\mathfrak{g}_2$ is isomorphic to $\mathbb{C}$, and $\mathfrak{g}_3 = 0$. Then the full Tanaka prolongation defines the simple $|2|$-graded complex Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

It is well known that simply connected indecomposable symmetric Riemannian manifolds (M, g) are in a one-to-one correspondence with simple $\mathbb{Z}_2$-graded Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ such that $\mathfrak{h} \subset \mathfrak{so}(n)$. If the symmetric space is quaternionic-Kählerian, then $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{f} \subset \mathfrak{so}(4k)$, where $n = 4k$, and $\mathfrak{f} \subset \mathfrak{sp}(k)$. The complexification of the algebra $\mathfrak{h} \oplus \mathbb{R}^{4k}$ coincides with $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k})$, where $\mathfrak{k} = \mathfrak{f} \otimes \mathbb{C} \subset \mathfrak{sp}(2k, \mathbb{C})$. Let e_1, e_2 be the standard basis of the space $\mathbb{C}^2$, and let

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We get the $\mathbb{Z}$-graded Lie algebra

$$\begin{aligned} \mathfrak{g} \otimes \mathbb{C} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\ &= \mathbb{C}F \oplus e_2 \otimes \mathbb{C}^{2k} \oplus (\mathfrak{k} \oplus \mathbb{C}H) \oplus e_1 \otimes \mathbb{C}^{2k} \oplus \mathbb{C}E. \end{aligned}$$

Conversely, each such $\mathbb{Z}$-graded Lie algebra defines (up to the duality) a simply connected quaternionic-Kählerian symmetric space. This completes the proof.

7. Construction of metrics and the classification theorem

Above we obtained a classification of weakly irreducible Berger algebras contained in $\mathfrak{sim}(n)$. In this section we will show that all these algebras can be realized as the holonomy algebras of Lorentzian manifolds and we will markedly simplify the construction of the metrics in [56]. We thereby complete the classification of holonomy algebras of Lorentzian manifolds.

The metrics realizing the Berger algebras of types 1 and 2 were constructed by Bérard-Bergery and Ikemakhen [21]. These metrics have the form

$$g = 2\,dv\,du + h + (\lambda v^2 + H_0)\,(du)^2,$$

where h is a Riemannian metric on $\mathbb{R}^n$ with the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$, $\lambda \in \mathbb{R}$, and H_0 is a sufficiently general function of the variables $x^1, \dots, x^n$. If $\lambda \neq 0$, then the holonomy algebra of this metric coincides with $\mathfrak{g}^{1,\mathfrak{h}}$; if $\lambda = 0$, then the holonomy algebra of the metric g coincides with $\mathfrak{g}^{2,\mathfrak{h}}$.

In [56] we gave a unified construction of metrics with all possible holonomy algebras. Here we simplify this construction.

Lemma 1. *For an arbitrary holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ of a Riemannian manifold there exists a $P \in \mathscr{P}(\mathfrak{h})$ such that the vector subspace $P(\mathbb{R}^n) \subset \mathfrak{h}$ generates the Lie algebra $\mathfrak{h}$.*

Proof. First we suppose that the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible. If $\mathfrak{h}$ is one of the holonomy algebras $\mathfrak{so}(n)$, $\mathfrak{u}(m)$, and $\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, then P can be taken to be one of the tensors described in § 5 for an arbitrary non-zero fixed $X \in \mathbb{R}^n$. It is obvious that $P(\mathbb{R}^n) \subset \mathfrak{h}$ generates the Lie algebra $\mathfrak{h}$. Similarly, if $\mathfrak{h} \subset \mathfrak{so}(n)$ is a symmetric Berger algebra, then we can consider a non-zero $X \in \mathbb{R}^n$ and put $P = R(X, \cdot)$, where R is the curvature tensor of the corresponding symmetric space. For $\mathfrak{su}(m)$ we use the isomorphism $\mathscr{P}(\mathfrak{su}(m)) \simeq (\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$ in § 5 and take P determined by an element $S \in (\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$ that does not belong to the space $(\odot^2(\mathbb{C}^{m_0})^* \otimes \mathbb{C}^{m_0})_0$ for any $m_0 < m$. We do the same for $\mathfrak{sp}(m)$.

The subalgebra $G_2 \subset \mathfrak{so}(7)$ is generated by the following matrices [15]:

$$\begin{aligned}
&A_1 = E_{12} - E_{34}, && A_2 = E_{12} - E_{56}, && A_3 = E_{13} + E_{24}, && A_4 = E_{13} - E_{67},\\
&A_5 = E_{14} - E_{23}, && A_6 = E_{14} - E_{57}, && A_7 = E_{15} + E_{26}, && A_8 = E_{15} + E_{47},\\
&A_9 = E_{16} - E_{25}, && A_{10} = E_{16} + E_{37}, && A_{11} = E_{17} - E_{36}, && A_{12} = E_{17} - E_{45},\\
&A_{13} = E_{27} - E_{35}, && A_{14} = E_{27} + E_{46},
\end{aligned}$$

where $E_{ij} \in \mathfrak{so}(7)$ $(i < j)$ is the skew-symmetric matrix with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

Consider the linear map $P \in \operatorname{Hom}(\mathbb{R}^7, G_2)$ given by the formulae

$$\begin{aligned}
&P(e_1) = A_6, && P(e_2) = A_4 + A_5, && P(e_3) = A_1 + A_7, && P(e_4) = A_1,\\
&P(e_5) = A_4, && P(e_6) = -A_5 + A_6, && P(e_7) = A_7.
\end{aligned}$$

With a computer it is easy to check that $P \in \mathscr{P}(G_2)$, and the elements $A_1, A_4, A_5, A_6, A_7 \in G_2$ generate the Lie algebra G_2.

The subalgebra $\mathfrak{spin}(7)\subset\mathfrak{so}(8)$ is generated by the following matrices [15]:

$$\begin{array}{llll}
A_1=E_{12}+E_{34}, & A_2=E_{13}-E_{24}, & A_3=E_{14}+E_{23}, & A_4=E_{56}+E_{78},\\
A_5=-E_{57}+E_{68}, & A_6=E_{58}+E_{67}, & A_7=-E_{15}+E_{26}, & A_8=E_{12}+E_{56},\\
A_9=E_{16}+E_{25}, & A_{10}=E_{37}-E_{48}, & A_{11}=E_{38}+E_{47}, & A_{12}=E_{17}+E_{28},\\
A_{13}=E_{18}-E_{27}, & A_{14}=E_{35}+E_{46}, & A_{15}=E_{36}-E_{45}, & A_{16}=E_{18}+E_{36},\\
A_{17}=E_{17}+E_{35}, & A_{18}=E_{26}-E_{48}, & A_{19}=E_{25}+E_{38}, & A_{20}=E_{23}+E_{67},\\
A_{21}=E_{24}+E_{57}. & & &
\end{array}$$

The linear map $P\in\mathrm{Hom}(\mathbb{R}^8,\mathfrak{spin}(7))$ defined by the formulae

$$\begin{array}{llll}
P(e_1)=0, & P(e_2)=-A_{14}, & P(e_3)=0, & P(e_4)=A_{21},\\
P(e_5)=A_{20}, & P(e_6)=A_{21}-A_{18}, & P(e_7)=A_{15}-A_{16}, & P(e_8)=A_{14}-A_{17},
\end{array}$$

belongs to the space $\mathscr{P}(\mathfrak{spin}(7))$, and the elements $A_{14}, A_{15}-A_{16}, A_{17}, A_{18}, A_{20}, A_{21}$ $\in\mathfrak{spin}(7)$ generate the Lie algebra $\mathfrak{spin}(7)$.

In the case of an arbitrary holonomy algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ the statement of the lemma follows from Theorem 14. □

Consider an arbitrary holonomy algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ of a Riemannian manifold. We will use the fact that $\mathfrak{h}$ is a weak Berger algebra, that is, $L(\mathscr{P}(\mathfrak{h}))=\mathfrak{h}$. The initial construction requires fixing a sufficient number of elements $P_1,\dots,P_N\in\mathscr{P}(\mathfrak{h})$ whose images generate $\mathfrak{h}$. The lemma just proved lets us confine ourselves to a single $P\in\mathscr{P}(\mathfrak{h})$. Recall that for $\mathfrak{h}$ the decompositions (4.4) and (4.5) hold. We will assume that the basis $e_1,\dots,e_n$ of the space $\mathbb{R}^n$ is compatible with the decomposition (4.4). Let $m_0=n_1+\dots+n_s=n-n_{s+1}$. Then $\mathfrak{h}\subset\mathfrak{so}(m_0)$, and $\mathfrak{h}$ does not annihilate any non-trivial subspace of $\mathbb{R}^{m_0}$. Note that in the case of the Lie algebra $\mathfrak{g}^{4,\mathfrak{h},m,\psi}$ we have $0<m_0\leqslant m$. We define the numbers P^k_{ji} such that $P(e_i)e_j=P^k_{ji}e_k$, and we consider the following metric on $\mathbb{R}^{n+2}$:

$$g=2\,dv\,du+\sum_{i=1}^n(dx^i)^2+2A_i\,dx^i\,du+H\cdot(du)^2, \tag{7.1}$$

where

$$A_i=\frac{1}{3}(P^i_{jk}+P^i_{kj})x^jx^k, \tag{7.2}$$

and H is a function that will depend on the type of the holonomy algebra that we want to obtain.

For the Lie algebra $\mathfrak{g}^{3,\mathfrak{h},\varphi}$ we define the numbers $\varphi_i=\varphi(P(e_i))$.

For the Lie algebra $\mathfrak{g}^{4,\mathfrak{h},m,\psi}$ we define the numbers ψ_{ij}, $j=m+1,\dots,n$, such that

$$\psi(P(e_i))=-\sum_{j=m+1}^n\psi_{ij}e_j. \tag{7.3}$$

Theorem 18. *The holonomy algebra $\mathfrak{g}$ of the metric g at the point 0 depends on the function H as follows*:

H	$\mathfrak{g}$
$v^2 + \sum_{i=m_0+1}^{n} (x^i)^2$	$\mathfrak{g}^{1,\mathfrak{h}}$
$\sum_{i=m_0+1}^{n} (x^i)^2$	$\mathfrak{g}^{2,\mathfrak{h}}$
$2v\varphi_i x^i + \sum_{i=m_0+1}^{n} (x^i)^2$	$\mathfrak{g}^{3,\mathfrak{h},\varphi}$
$2\sum_{j=m+1}^{n} \psi_{ij} x^i x^j + \sum_{i=m_0+1}^{m} (x^i)^2$	$\mathfrak{g}^{4,\mathfrak{h},m,\psi}$

From Theorems 16 and 18 we get the main classification theorem.

Theorem 19. *A subalgebra $\mathfrak{g} \subset \mathfrak{so}(1,n+1)$ is a weakly irreducible not irreducible holonomy algebra of a Lorentzian manifold if and only if $\mathfrak{g}$ is conjugate to one of the subalgebras $\mathfrak{g}^{1,\mathfrak{h}}$, $\mathfrak{g}^{2,\mathfrak{h}}$, $\mathfrak{g}^{3,\mathfrak{h},\varphi}$, and $\mathfrak{g}^{4,\mathfrak{h},m,\psi} \subset \mathfrak{sim}(n)$, where $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.*

Proof of Theorem 18. Consider the field of frames (4.15). Let $X_p = p$ and $X_q = q$. The indices $a, b, c, \dots$ will run through all the indices of the basis vector fields. The components of the connection Γ^c_{ba} are defined by the formula $\nabla_{X_a} X_b = \Gamma^c_{ba} X_c$. The constructed metrics are analytic. From the proof of Theorem 9.2 in [94] it follows that $\mathfrak{g}$ is generated by the elements of the form

$$\nabla_{X_{a_\alpha}} \cdots \nabla_{X_{a_1}} R(X_a, X_b)(0) \in \mathfrak{so}(T_0M, g_0) = \mathfrak{so}(1, n+1), \qquad \alpha = 0, 1, 2, \dots,$$

where ∇ is the Levi-Civita connection defined by the metric g, and R is the curvature tensor. The components of the curvature tensor are defined by

$$R(X_a, X_b)X_c = \sum_d R^d_{cab} X_d.$$

We have the recursion formula

$$\begin{aligned} \nabla_{a_\alpha} \cdots \nabla_{a_1} R^d_{cab} = X_{a_\alpha} \nabla_{a_{\alpha-1}} \cdots \nabla_{a_1} R^d_{cab} \\ + [\Gamma_{a_\alpha}, \nabla_{X_{a_{\alpha-1}}} \cdots \nabla_{X_{a_1}} R(X_a, X_b)]^d_c, \end{aligned} \tag{7.4}$$

where Γ_{a_α} denotes the operator with matrix $(\Gamma^a_{ba_\alpha})$. Since we consider the Walker metric, we have $\mathfrak{g} \subset \mathfrak{sim}(n)$.

From the foregoing it is not hard to find the holonomy algebra $\mathfrak{g}$. We go through the computations for an algebra of the fourth type. The proof for other types is similar. Let $H = 2\sum_{j=m+1}^{n} \psi_{ij} x^i x^j + \sum_{i=m_0+1}^{m} (x^i)^2$. We must prove the equality $\mathfrak{g} = \mathfrak{g}^{4,\mathfrak{h},m,\psi}$. It is clear that $\nabla \partial_v = 0$, and hence $\mathfrak{g} \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$.

The possibly non-zero Lie brackets of the basis vector fields are the following:

$$[X_i,X_j]=-F_{ij}p=2P^j_{ik}x^kp,\qquad [X_i,q]=C^p_{iq}p,$$

$$C^p_{iq}=-\frac{1}{2}\partial_iH=\begin{cases}-\sum\limits_{j=m+1}^{n}\psi_{ij}x^j, & 1\leqslant i\leqslant m_0,\\ -x^i, & m_0+1\leqslant i\leqslant m,\\ -\psi_{ki}x^k, & m+1\leqslant i\leqslant n.\end{cases}$$

Using this, it is easy to find the matrices of the operators Γ_a, namely, $\Gamma_p=0$,

$$\Gamma_k=\begin{pmatrix}0 & Y^t_k & 0\\ 0 & 0 & -Y_k\\ 0 & 0 & 0\end{pmatrix},\qquad Y^t_k=(P^k_{1i}x^i,\dots,P^k_{m_0i}x^i,0,\dots,0),$$

$$\Gamma_q=\begin{pmatrix}0 & Z^t & 0\\ 0 & (P^i_{jk}x^k) & -Z\\ 0 & 0 & 0\end{pmatrix},\qquad Z^t=-(C^p_{1q},\dots,C^p_{nq}).$$

It suffices to compute the following components of the curvature tensor:

$$R^k_{jiq}=P^k_{ji},\quad R^k_{jil}=0,\quad R^k_{qij}=-P^i_{jk},$$
$$R^j_{qjq}=-1,\quad m_0+1\leqslant j\leqslant m,\qquad R^l_{qjq}=-\psi_{jl},\quad m+1\leqslant l\leqslant n.$$

This implies that

$$\mathrm{pr}_{\mathfrak{so}(n)}\big(R(X_i,q)(0)\big)=P(e_i),\quad \mathrm{pr}_{\mathbb{R}^n}\big(R(X_i,q)(0)\big)=\psi(P(e_i)),$$
$$\mathrm{pr}_{\mathbb{R}^n}\big(R(X_j,q)(0)\big)=-e_j,\qquad m_0+1\leqslant j\leqslant m,$$
$$\mathrm{pr}_{\mathbb{R}^n}\big(R(X_i,X_j)(0)\big)=P(e_j)e_i-P(e_i)e_j.$$

We get the inclusion $\mathfrak{g}^{4,\mathfrak{h},m,\psi}\subset\mathfrak{g}$. The formula (7.4) and induction enable us to get the reverse inclusion. $\square$

Let us consider two *examples*. From the proof of Lemma 1 it follows that the holonomy algebra of the metric

$$g=2\,dv\,du+\sum_{i=1}^{7}(dx^i)^2+2\sum_{i=1}^{7}A_i\,dx^i\,du,$$

where

$$A_1 = \frac{2}{3}(2x^2x^3 + x^1x^4 + 2x^2x^4 + 2x^3x^5 + x^5x^7),$$
$$A_2 = \frac{2}{3}(-x^1x^3 - x^2x^3 - x^1x^4 + 2x^3x^6 + x^6x^7),$$
$$A_3 = \frac{2}{3}(-x^1x^2 + (x^2)^2 - x^3x^4 - (x^4)^2 - x^1x^5 - x^2x^6),$$
$$A_4 = \frac{2}{3}(-(x^1)^2 - x^1x^2 + (x^3)^2 + x^3x^4),$$
$$A_5 = \frac{2}{3}(-x^1x^3 - 2x^1x^7 - x^6x^7),$$
$$A_6 = \frac{2}{3}(-x^2x^3 - 2x^2x^7 - x^5x^7),$$
$$A_7 = \frac{2}{3}(x^1x^5 + x^2x^6 + 2x^5x^6),$$

at the point $0 \in \mathbb{R}^9$ coincides with $\mathfrak{g}^{2,G_2} \subset \mathfrak{so}(1,8)$. Similarly, the holonomy algebra of the metric

$$g = 2\,dv\,du + \sum_{i=1}^{8}(dx^i)^2 + 2\sum_{i=1}^{8} A_i\,dx^i\,du,$$

where

$$A_1 = -\frac{4}{3}x^7x^8, \qquad A_2 = \frac{2}{3}((x^4)^2 + x^3x^5 + x^4x^6 - (x^6)^2),$$
$$A_3 = -\frac{4}{3}x^2x^5, \qquad A_4 = \frac{2}{3}(-x^2x^4 - 2x^2x^6 - x^5x^7 + 2x^6x^8),$$
$$A_5 = \frac{2}{3}(x^2x^3 + 2x^4x^7 + x^6x^7), \qquad A_6 = \frac{2}{3}(x^2x^4 + x^2x^6 + x^5x^7 - x^4x^8),$$
$$A_7 = \frac{2}{3}(-x^4x^5 - 2x^5x^6 + x^1x^8), \qquad A_8 = \frac{2}{3}(-x^4x^6 + x^1x^7),$$

at the point $0 \in \mathbb{R}^{10}$ coincides with $\mathfrak{g}^{2,\mathfrak{spin}(7)} \subset \mathfrak{so}(1,9)$.

8. The Einstein equation

In this section we consider the connection between holonomy algebras and the Einstein equation. We will find the holonomy algebras of Einstein Lorentzian manifolds. Then we will show that in the case of a non-zero cosmological constant, there exist special coordinates on a Walker manifold enabling us to essentially simplify the Einstein equation. Examples of Einstein metrics will be given. This topic is motivated by the paper [76] by the theoretical physicists Gibbons and Pope. The results of this section were published in [60], [61], [62], [74].

8.1. Holonomy algebras of Einstein Lorentzian manifolds. Let (M,g) be a Lorentzian manifold with holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$. First of all, the following theorem was proved in [72].

Theorem 20. *Let (M,g) be a locally indecomposable Einstein Lorentzian manifold admitting a parallel distribution of isotropic lines. Then the holonomy of (M,g) is* *of type* 1 *or* 2*. If the cosmological constant of* (M, g) *is non-zero, then the holonomy algebra of* (M, g) *is of type* 1*. If* (M, g) *admits locally a parallel isotropic vector field, then* (M, g) *is Ricci-flat.*

The following two theorems from [60] complete the classification.

Theorem 21. *Let* (M, g) *be a locally indecomposable* $(n+2)$*-dimensional Lorentzian manifold admitting a parallel distribution of isotropic lines. If* (M, g) *is Ricci-flat, then one of the following statements holds.*

(I) *The holonomy algebra* $\mathfrak{g}$ *of* (M, g) *is of type* 1*, and in the decomposition* (4.5) *for* $\mathfrak{h} \subset \mathfrak{so}(n)$ *at least one of the subalgebras* $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ *coincides with one of the Lie algebras* $\mathfrak{so}(n_i)$*,* $\mathfrak{u}(n_i/2)$*, or* $\mathfrak{sp}(n_i/4) \oplus \mathfrak{sp}(1)$*, or with a symmetric Berger algebra.*

(II) *The holonomy algebra* $\mathfrak{g}$ *of* (M, g) *is of type* 2*, and in the decomposition* (4.5) *for* $\mathfrak{h} \subset \mathfrak{so}(n)$ *each subalgebra* $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ *coincides with one of the Lie algebras* $\mathfrak{so}(n_i)$*,* $\mathfrak{su}(n_i/2)$*,* $\mathfrak{sp}(n_i/4)$*,* $G_2 \subset \mathfrak{so}(7)$*, or* $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$*.*

Theorem 22. *Let* (M, g) *be a locally indecomposable* $(n+2)$*-dimensional Lorentzian manifold admitting a parallel distribution of isotropic lines. If* (M, g) *is Einstein and not Ricci-flat, then the holonomy algebra* $\mathfrak{g}$ *of* (M, g) *is of type* 1*, and in the decomposition* (4.5) *for* $\mathfrak{h} \subset \mathfrak{so}(n)$ *each subalgebra* $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ *coincides with one of the Lie algebras* $\mathfrak{so}(n_i)$*,* $\mathfrak{u}(n_i/2)$*, or* $\mathfrak{sp}(n_i/4) \oplus \mathfrak{sp}(1)$*, or with a symmetric Berger algebra. Moreover,* $n_{s+1} = 0$*.*

8.2. Examples of Einstein metrics. In this section we show the existence of metrics for each holonomy algebra obtained in the previous section.

From (4.7) and (4.8) it follows that the Einstein equation

$$\mathrm{Ric} = \Lambda g$$

for the metric (4.11) can be rewritten in the notation of § 4.2 as

$$\lambda = \Lambda, \quad \mathrm{Ric}(h) = \Lambda h, \quad \vec{v} = \widetilde{\mathrm{Ric}}(P), \quad \mathrm{tr}\, T = 0. \tag{8.1}$$

First of all, consider the metric (4.11) such that h is an Einstein Riemannian metric with the holonomy algebra $\mathfrak{h}$ and the non-zero cosmological constant Λ, and $A = 0$. Let

$$H = \Lambda v^2 + H_0,$$

where H_0 is a function depending on the coordinates $x^1, \dots, x^n$. Then the first three equations in (8.1) are satisfied. From (4.18) it follows that the last equation has the form

$$\Delta H_0 = 0,$$

where

$$\Delta = h^{ij}(\partial_i \partial_j - \Gamma^k_{ij} \partial_k) \tag{8.2}$$

is the Laplace–Beltrami operator of the metric h. Choosing a sufficiently general harmonic function H_0, we get that g is an Einstein metric and is indecomposable. From Theorem 22 it follows that $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$.

Choosing $\Lambda = 0$ in the same construction, we get a Ricci-flat metric with the holonomy algebra $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$.

Let us construct a Ricci-flat metric with the holonomy algebra $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$, where $\mathfrak{h}$ is as in part (I) of Theorem 21. To do this we use the construction of § 7. Consider a $P \in \mathscr{P}(\mathfrak{h})$ with $\widetilde{\mathrm{Ric}}(P) \neq 0$. Recall that $h_{ij} = \delta_{ij}$. We let

$$H = vH_1 + H_0,$$

where H_1 and H_0 are functions of the coordinates $x^1, \dots, x^n$. The third equation in (8.1) takes the form

$$\partial_k H_1 = 2 \sum_i P^k_{ii},$$

and hence it suffices to take

$$H_1 = 2 \sum_{i,k} P^k_{ii} x^k.$$

The last equation has the form

$$\frac{1}{2} \sum_i \partial_i^2 H_0 - \frac{1}{4} \sum_{i,j} F_{ij}^2 - \frac{1}{2} H_1 \sum_i \partial_i A_i - 2A_i \sum_k P^i_{kk} = 0.$$

Note that

$$F_{ij} = 2P^j_{ik} x^k, \qquad \sum_i \partial_i A_i = -2 \sum_{i,k} P^k_{ii} x^k.$$

We get an equation of the form $\sum_i \partial_i^2 H_0 = K$, where K is a polynomial of degree two. A particular solution of this equation can be found in the form

$$H_0 = \frac{1}{2}(x^1)^2 K_2 + \frac{1}{6}(x^1)^3 \partial_1 K_1 + \frac{1}{24}(x^i)^4 (\partial_i)^2 K,$$

where

$$K_1 = K - \frac{1}{2}(x^i)^2 (\partial_i)^2 K, \qquad K_2 = K_1 - x^1 \partial_1 K_1.$$

In order to make the metric g indecomposable it suffices to add the harmonic function

$$(x^1)^2 + \dots + (x^{n-1})^2 - (n-1)(x^n)^2$$

to the function H_0 obtained. Since $\partial_v \partial_i H \neq 0$, the holonomy algebra of the metric g is of type 1 or 3. From Theorem 21 it follows that $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$.

It is possible to construct in a similar way an example of a Ricci-flat metric with the holonomy algebra $\mathfrak{h} \ltimes \mathbb{R}^n$, where $\mathfrak{h}$ is as in part (II) of Theorem 21. To this end, it suffices to consider a $P \in \mathscr{P}(\mathfrak{h})$ with $\widetilde{\mathrm{Ric}}(P) = 0$, take $H_1 = 0$, and thus obtain the required H_0.

We have proved the following theorem.

Theorem 23. *Let $\mathfrak{g}$ be an algebra as in Theorem 21 or 22. Then there exists an $(n+2)$-dimensional Einstein (or Ricci-flat) Lorentzian manifold with the holonomy algebra $\mathfrak{g}$.*

Example 1. In § 7 we constructed metrics with the holonomy algebras

$$\mathfrak{g}^{2,G_2} \subset \mathfrak{so}(1,8) \quad \text{and} \quad \mathfrak{g}^{2,\mathfrak{spin}(7)} \subset \mathfrak{so}(1,9).$$

Choosing the function H in the way just described, we get Ricci-flat metrics with the same holonomy algebras.

8.3. Lorentzian manifolds with totally isotropic Ricci operator. In the last section we saw that, unlike the case of Riemannian manifolds, Lorentzian manifolds with any of the holonomy algebras are not automatically Ricci-flat or Einstein. Now we will see that, nevertheless, Lorentzian manifolds with certain holonomy algebras automatically satisfy a weaker condition on the Ricci tensor.

A Lorentzian manifold (M,g) is said to be *totally Ricci-isotropic* if the image of its Ricci operator is isotropic, that is,

$$g\bigl(\mathrm{Ric}(X),\mathrm{Ric}(Y)\bigr)=0$$

for all vector fields X and Y. Obviously, any Ricci-flat Lorentzian manifold is totally Ricci-isotropic. If (M,g) is a spin manifold and admits a parallel spinor field, then it is totally Ricci-isotropic [40], [52].

Theorem 24. *Let (M,g) be a locally indecomposable $(n+2)$-dimensional Lorentzian manifold admitting a parallel distribution of isotropic lines. If (M,g) is totally Ricci-isotropic, then its holonomy algebra is the same as in Theorem 21.*

Theorem 25. *Let (M,g) be a locally indecomposable $(n+2)$-dimensional Lorentzian manifold admitting a parallel distribution of isotropic lines. If the holonomy algebra of (M,g) is of type 2 and in the decomposition (4.5) of the algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ each subalgebra $\mathfrak{h}_i\subset\mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{su}(n_i/2)$, $\mathfrak{sp}(n_i/4)$, $G_2\subset\mathfrak{so}(7)$, or $\mathfrak{spin}(7)\subset\mathfrak{so}(8)$, then the manifold (M,g) is totally Ricci-isotropic.*

Note that this theorem can be also proved by the following argument. Locally, (M,g) admits a spin structure. From [72] and [100] it follows that (M,g) admits locally parallel spinor fields and hence is totally Ricci-isotropic.

8.4. Simplification of the Einstein equation. The Einstein equation for the metric (4.11) was considered by Gibbons and Pope [76]. First of all, it implies that

$$H=\Lambda v^2+vH_1+H_0,\qquad \partial_vH_1=\partial_vH_0=0.$$

Further, it is equivalent to the system of equations

$$\Delta H_0-\frac{1}{2}F^{ij}F_{ij}-2A^i\partial_iH_1-H_1\nabla^iA_i+2\Lambda A^iA_i-2\nabla^i\dot A_i \\ +\frac{1}{2}\dot h^{ij}\dot h_{ij}+h^{ij}\ddot h_{ij}+\frac{1}{2}h^{ij}\dot h_{ij}H_1=0, \tag{8.3}$$

$$\nabla^jF_{ij}+\partial_iH_1-2\Lambda A_i+\nabla^j\dot h_{ij}-\partial_i(h^{jk}\dot h_{jk})=0, \tag{8.4}$$

$$\Delta H_1-2\Lambda\nabla^iA_i-\Lambda h^{ij}\dot h_{ij}=0, \tag{8.5}$$

$$\mathrm{Ric}_{ij}=\Lambda h_{ij}, \tag{8.6}$$

where the operator Δ is given by the formula (8.2). These equations can be obtained by considering the equations (8.1) and applying the formulae in § 4.2.

The Walker coordinates are not defined uniquely. For instance, Schimming [110] showed that if $\partial_vH=0$, then the coordinates can be chosen in such a way that $A=0$ and $H=0$. The main theorem of this subsection makes it possible to find similar coordinates and thereby to simplify the Einstein equation for the case $\Lambda\neq0$.

Theorem 26. *Let (M,g) be a Lorentzian manifold of dimension $n+2$ admitting a parallel distribution of isotropic lines. If (M,g) is Einstein with a non-zero cosmological constant Λ, then there exist local coordinates $v, x^1, \dots, x^n, u$ such that the metric g has the form*

$$g = 2\,dv\,du + h + (\Lambda v^2 + H_0)\,(du)^2,$$

where $\partial_v H_0 = 0$, and h is in the u-family of Einstein Riemannian metrics with the cosmological constant Λ and satisfying the equations

$$\Delta H_0 + \frac{1}{2} h^{ij} \ddot{h}_{ij} = 0, \tag{8.7}$$

$$\nabla^j \dot{h}_{ij} = 0, \tag{8.8}$$

$$h^{ij} \dot{h}_{ij} = 0, \tag{8.9}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}. \tag{8.10}$$

Conversely, any such metric is Einstein.

Thus, we have reduced the Einstein equation with $\Lambda \neq 0$ for Lorentzian metrics to the problem of finding families of Einstein Riemannian metrics satisfying the equations (8.8) and (8.9) together with a function H_0 satisfying the equation (8.7).

8.5. The case of dimension 4. Let us consider the case of dimension 4, that is, $n = 2$. We write $x = x^1$, $y = x^2$.

Ricci-flat Walker metrics in dimension 4 are found in [92]. They are given by $h = (dx)^2 + (dy)^2$, $A_2 = 0$, and $H = -(\partial_x A_1)v + H_0$, where A_1 is a harmonic function and H_0 is a solution of the Poisson equation.

In [102] all 4-dimensional Einstein Walker metrics with $\Lambda \neq 0$ are described. The coordinates can be chosen in such a way that h is a metric of constant sectional curvature independent of u. Furthermore, $A = W\,dz + \overline{W}\,d\overline{z}$ and $W = i\partial_z L$, where $z = x + iy$, and L is the $\mathbb{R}$-valued function given by the formula

$$L = 2\,\mathrm{Re}\left(\phi \partial_z(\log P_0) - \frac{1}{2}\partial_z \phi\right), \qquad 2P_0^2 = \left(1 + \frac{\Lambda}{|\Lambda|} z\overline{z}\right)^2, \tag{8.11}$$

with $\phi = \phi(z,u)$ an arbitrary function holomorphic with respect to z and smooth with respect to u. Finally, $H = \Lambda^2 v + H_0$, and the function $H_0 = H_0(z, \overline{z}, u)$ can be found similarly.

In this section we give examples of Einstein Walker metrics with $\Lambda \neq 0$ such that $A = 0$, and h depends on u. The solutions in [102] are not useful for constructing examples of such form, since 'simple' functions $\phi(z,u)$ determine 'complicated' forms A. Similar examples can be constructed in dimension 5, a case discussed in [76], [78].

We note that in dimension 2 (and 3) any Einstein Riemannian metric has constant sectional curvature, and hence such metrics with the same Λ are locally isometric, and the coordinates can be chosen so that $\partial_u h = 0$. As in [102], it is not hard to show that if $\Lambda > 0$, then we can assume that $h = \big((dx)^2 + \sin^2 x\,(dy)^2\big)/\Lambda$

and $H = \Lambda v^2 + H_0$, and the Einstein equation reduces to the system

$$\Delta_{S^2} f = -2f, \qquad \Delta_{S^2} H_0 = 2\Lambda\biggl(2f^2 - (\partial_x f)^2 + \frac{(\partial_y f)^2}{\sin^2 x}\biggr), \\ \Delta_{S^2} = \partial_x^2 + \frac{\partial_y^2}{\sin^2 x} + \cot x\,\partial_x. \tag{8.12}$$

The function f determines the 1-form

$$A = -\frac{\partial_y f}{\sin x}\,dx + \sin x\,\partial_x f\,dy.$$

Similarly, if $\Lambda < 0$, then we consider

$$h = \frac{1}{-\Lambda\cdot x^2}\bigl((dx)^2 + (dy)^2\bigr)$$

and get that

$$\Delta_{L^2} f = 2f, \qquad \Delta_{L^2} H_0 = -4\Lambda f^2 - 2\Lambda x^2\bigl((\partial_x f)^2 + (\partial_y f)^2\bigr), \\ \Delta_{L^2} = x^2(\partial_x^2 + \partial_y^2), \tag{8.13}$$

and $A = -\partial_y f\,dx + \partial_x f\,dy$. Thus, in order to find particular solutions of the system of equations (8.7)–(8.10), it is convenient first to find f and then to get rid of the 1-form A by changing the coordinates. After such a coordinate change, the metric h does not depend on u if and only if A is the Killing form for h [76]. If $\Lambda > 0$, then this occurs if and only if

$$f = c_1(u)\sin x\sin y + c_2(u)\sin x\cos y + c_3(u)\cos x,$$

and for $\Lambda < 0$ this is equivalent to the equality

$$f = c_1(u)\frac{1}{x} + c_2(u)\frac{y}{x} + c_3(u)\frac{x^2+y^2}{x}\,.$$

The functions $\phi(z,u) = c(u)$, $c(u)z$, and $c(u)z^2$ in (8.11) determine the Killing form A [74]. For other functions ϕ the form A has a complicated structure. Let g be an Einstein metric of the form (4.11) with $\Lambda \neq 0$, $A = 0$, and $H = \Lambda v^2 + H_0$. The curvature tensor R of the metric g has the form

$$R(p,q) = \Lambda p\wedge q, \quad R(X,Y) = \Lambda X\wedge Y, \quad R(X,q) = -p\wedge T(X), \quad R(p,X) = 0.$$

The metric g is indecomposable if and only if $T \neq 0$. In this case the holonomy algebra coincides with $\mathfrak{sim}(2)$.

For the Weyl tensor we have

$$W(p,q) = \frac{\Lambda}{3}p\wedge q, \qquad W(p,X) = -\frac{2\Lambda}{3}p\wedge X, \\ W(X,Y) = \frac{\Lambda}{3}X\wedge Y, \qquad W(X,q) = -\frac{2\Lambda}{3}X\wedge q - p\wedge T(X).$$

In [81] it is shown that the Petrov type of the metric g is either II or D (and it can change from point to point). From the Bel criterion it follows that g is of type II at a point $m \in M$ if and only if $T_m \neq 0$, otherwise g is of type D. Since the endomorphism T_m is symmetric and trace-free, it either is zero or has rank 2. Consequently, $T_m = 0$ if and only if $\det T_m = 0$.

Example 2. Consider the function $f = c(u)x^2$. Then $A = 2xc(u)\,dy$. Choose $H_0 = -\Lambda x^4 c^2(u)$. To get rid of the form A, we solve the system of equations

$$\frac{dx(u)}{du} = 0, \qquad \frac{dy(u)}{du} = 2\Lambda c(u) x^3(u)$$

with the initial data $x(0) = \widetilde{x}$ and $y(0) = \widetilde{y}$. We get the transformation

$$v = \widetilde{v}, \quad x = \widetilde{x}, \quad y = \widetilde{y} + 2\Lambda b(u)\widetilde{x}^{\,3}, \quad u = \widetilde{u},$$

where the function $b(u)$ satisfies $db(u)/du = c(u)$ and $b(0) = 0$. With respect to the new coordinates

$$g = 2\,dv\,du + h(u) + \left(\Lambda v^2 + 3\Lambda x^4 c^2(u)\right)(du)^2,$$
$$h(u) = \frac{1}{-\Lambda \cdot x^2}\left(\left(36\Lambda^2 b^2(u) x^4 + 1\right)(dx)^2 + 12\Lambda b(u) x^2\,dx\,dy + (dy)^2\right).$$

Let $c(u) \equiv 1$. Then $b(u) = u$ and $\det T = -9\Lambda^4 x^4 (x^4 + v^2)$. The equality $\det T_m = 0$ ($m = (v, x, y, u)$) is equivalent to $v = 0$. The metric g is indecomposable. This metric is of Petrov type D on the set $\{(0, x, y, u)\}$ and of type II on its complement.

Example 3. The function $f = \log\bigl(\tan(x/2)\bigr)\cos x + 1$ is a particular solution of the first equation in (8.12). We get that $A = \bigl(\cos x - \log(\cot(x/2))\sin^2 x\bigr)\,dy$. Consider the transformation

$$\widetilde{v} = v, \quad \widetilde{x} = x, \quad \widetilde{y} = y - \Lambda u\left(\log\left(\tan\frac{x}{2}\right) - \frac{\cos x}{\sin^2 x}\right), \quad \widetilde{u} = u.$$

With respect to the new coordinates we have

$$g = 2\,dv\,du + h(u) + (\Lambda v^2 + \widetilde{H}_0)(du)^2,$$
$$h(u) = \left(\frac{1}{\Lambda} + \frac{4\Lambda u^2}{\sin^4 x}\right)(dx)^2 + \frac{4u}{\sin x}\,dx\,dy + \frac{\sin^2 x}{\Lambda}(dy)^2,$$

where $\widetilde{H}_0$ satisfies the equation $\Delta_h \widetilde{H}_0 = -\frac{1}{2} h^{ij}\ddot{h}_{ij}$. One example of such a function $\widetilde{H}_0$ is

$$\widetilde{H}_0 = -\Lambda\left(\frac{1}{\sin^2 x} + \log^2\left(\cot\frac{x}{2}\right)\right).$$

We have

$$\det T = -\frac{\Lambda^4}{\sin^4 x}\left(v^2 + \left(\log\left(\cot\frac{x}{2}\right)\cos x - 1\right)^2\right).$$

Hence, the metric g is of Petrov type D on the set

$$\left\{(0, x, y, u) \,\middle|\, \log\left(\cot\frac{x}{2}\right)\cos x - 1 = 0\right\}$$

and of type II on the complement of this set. The metric is indecomposable.

9. Riemannian and Lorentzian manifolds with recurrent spinor fields

Let (M, g) be a pseudo-Riemannian spin manifold with signature (r, s), and S the corresponding complex spinor bundle with the induced connection ∇^S. A spinor field $s \in \Gamma(S)$ is said to be *recurrent* if

$$\nabla^S_X s = \theta(X)s \tag{9.1}$$

for all vector fields $X \in \Gamma(TM)$ (here θ is a complex-valued 1-form). If $\theta = 0$, then s is *a parallel* spinor field. For a recurrent spinor field s there exists a locally defined non-vanishing function f such that the field fs is parallel if and only if $d\theta = 0$. If the manifold M is simply connected, then such a function is defined globally.

The study of Riemannian spin manifolds carrying parallel spinor fields was initiated by Hitchin [83] and then continued by Friedrich [54]. Wang characterized simply connected Riemannian spin manifolds admitting parallel spinor fields in terms of their holonomy groups [121]. A similar result was obtained by Leistner for Lorentzian manifolds [98], [99], by Baum and Kath for pseudo-Riemannian manifolds with irreducible holonomy groups [15], and by Ikemakhen in the case of pseudo-Riemannian manifolds with neutral signature (n, n) that admit two mutually complementary parallel isotropic distributions [86].

Friedrich [54] considered equation (9.1) on a Riemannian spin manifold assuming that θ is a real-valued 1-form. He proved that this equation implies that the Ricci tensor is zero and $d\theta = 0$. Below we will see that this statement does not hold for Lorentzian manifolds. Example 1 in [54] provides a solution s to equation (9.1) with $\theta = i\omega$ and $d\omega \neq 0$ for a real-valued 1-form ω on the compact Riemannian manifold (M, g) that is the product of the non-flat torus T^2 and the circle S^1. In fact, the recurrent spinor field s comes from a locally defined recurrent spinor field on the non-Ricci-flat Kähler manifold T^2. The existence of the last spinor field is shown in Theorem 27 below.

The spinor bundle S of a pseudo-Riemannian manifold (M, g) admits a parallel 1-dimensional complex subbundle if and only if (M, g) admits non-vanishing recurrent spinor fields in a neighbourhood of each point such that these fields are proportional on the intersections of their domains. In the present section we study some classes of pseudo-Riemannian spin manifolds (M, g) whose spinor bundles admit parallel 1-dimensional complex subbundles.

9.1. Riemannian manifolds. Wang [121] showed that a simply connected locally indecomposable Riemannian manifold (M, g) admits a parallel spinor field if and only if its holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is one of $\mathfrak{su}(n/2)$, $\mathfrak{sp}(n/4)$, G_2, and $\mathfrak{spin}(7)$.

In [67] the following results for Riemannian manifolds with recurrent spinor fields were obtained.

Theorem 27. *Let (M, g) be a locally indecomposable n-dimensional simply connected Riemannian spin manifold. Then its spinor bundle S admits a parallel 1-dimensional complex subbundle if and only if either the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ of (M, g) is one of $\mathfrak{u}(n/2)$, $\mathfrak{su}(n/2)$, $\mathfrak{sp}(n/4)$, $G_2 \subset \mathfrak{so}(7)$, and $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, or (M, g) is a locally symmetric Kählerian manifold.*

Corollary 4. *Let (M, g) be a simply connected Riemannian spin manifold with irreducible holonomy algebra and without non-zero parallel spinor fields. Then the* *spinor bundle S admits a parallel 1-dimensional complex subbundle if and only if (M,g) is a Kählerian manifold and is not Ricci-flat.*

Corollary 5. *Let (M,g) be a simply connected complete Riemannian spin manifold without non-zero parallel spinor fields and with holonomy algebra not irreducible. Then its spinor bundle S admits a parallel 1-dimensional complex subbundle if and only if (M,g) is a direct product of a Kählerian not Ricci-flat spin manifold and a Riemannian spin manifold with a non-zero parallel spinor field.*

Theorem 28. *Let (M,g) be a locally indecomposable n-dimensional simply connected Kählerian spin manifold that is not Ricci-flat. Then its spinor bundle S admits exactly two parallel 1-dimensional complex subbundles.*

9.2. Lorentzian manifolds. The holonomy algebras of Lorentzian spin manifolds admitting non-zero parallel spinor fields are classified in [98], [99]. We suppose now that the spinor bundle of (M,g) admits a parallel 1-dimensional complex subbundle and (M,g) does not admit any non-zero parallel spinor field.

Theorem 29. *Let (M,g) be a simply connected complete Lorentzian spin manifold. Suppose that (M,g) does not admit a non-zero parallel spinor field. In this case the spinor bundle S admits a parallel 1-dimensional complex subbundle if and only if one of the following conditions holds*:

1) *(M,g) is a direct product of $(\mathbb{R},-(dt)^2)$ and a Riemannian spin manifold (N,h) such that the spinor bundle of (N,h) admits a parallel 1-dimensional complex subbundle and (N,h) does not admit any non-zero parallel spinor field*;

2) *(M,g) is a direct product of an indecomposable Lorentzian spin manifold and a Riemannian spin manifold (N,h) such that the spinor bundles of both manifolds admit parallel 1-dimensional complex subbundles and at least one of these manifolds does not admit any non-zero parallel spinor field.*

Let us consider a locally indecomposable Lorentzian manifold (M,g). Suppose that the spinor bundle of (M,g) admits a parallel 1-dimensional complex subbundle l. Let $s\in\Gamma(l)$ be any local non-vanishing section of the bundle l. Let $p\in\Gamma(TM)$ be the corresponding Dirac current. The vector field p is defined by the equality

$$g(p,X)=-\langle X\cdot s,s\rangle,$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the Hermitian product on S. It turns out that p is a recurrent vector field. In the case of Lorentzian manifolds, the Dirac current satisfies $g(p,p)\leqslant 0$, and the zeros of the field p coincide with the zeros of the field s. Since s is non-vanishing and p is a recurrent field, either $g(p,p)<0$ or $g(p,p)=0$. In the first case the manifold is decomposable. Thus, we get that p is an isotropic recurrent vector field, and the manifold (M,g) admits a parallel distribution of isotropic lines, that is, its holonomy algebra is contained in $\mathfrak{sim}(n)$.

In [98], [99] it was shown that (M,g) admits a parallel spinor field if and only if $\mathfrak{g}=\mathfrak{g}^{2,\mathfrak{h}}=\mathfrak{h}\ltimes\mathbb{R}^n$ and in the decomposition (4.5) for the subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ each of the subalgebras $\mathfrak{h}_i\subset\mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{su}(n_i/2)$, $\mathfrak{sp}(n_i/4)$, $G_2\subset\mathfrak{so}(7)$, and $\mathfrak{spin}(7)\subset\mathfrak{so}(8)$.

In [67] we prove the following theorem.

Theorem 30. *Let (M,g) be a simply connected locally indecomposable $(n+2)$-dimensional Lorentzian spin manifold. Then its spinor bundle S admits a parallel 1-dimensional complex subbundle if and only if (M,g) admits a parallel distribution of isotropic lines (that is, its holonomy algebra $\mathfrak{g}$ is contained in $\mathfrak{sim}(n)$), and in the decomposition (4.5) for the subalgebra $\mathfrak{h}=\mathrm{pr}_{\mathfrak{so}(n)}\,\mathfrak{g}$ each of the subalgebras $\mathfrak{h}_i\subset\mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{u}(n_i/2)$, $\mathfrak{su}(n_i/2)$, $\mathfrak{sp}(n_i/4)$, G_2, and $\mathfrak{spin}(7)$, or with the holonomy algebra of an indecomposable Kählerian symmetric space. The number of parallel 1-dimensional complex subbundles of S equals the number of 1-dimensional complex subspaces of the spin module Δ_n that are preserved by the algebra $\mathfrak{h}$.*

10. Conformally flat Lorentzian manifolds with special holonomy groups

In this section we will give a local classification of conformally flat Lorentzian manifolds with special holonomy groups. The corresponding local metrics are certain extensions of Riemannian metrics of constant sectional curvature to Walker metrics. This result was published in [64], [66].

Kurita [96] proved that a conformally flat Riemannian manifold is a product of two spaces of constant sectional curvature, or it is the product of a space of constant sectional curvature and an interval, or its restricted holonomy group is the connected component of the identity of the orthogonal group. The last condition represents the most general situation, and among various manifolds satisfying the last condition one should emphasize only the spaces of constant sectional curvature. It is clear that there are no conformally flat Riemannian manifolds with special holonomy groups.

In [66] we generalized the Kurita theorem to the case of pseudo-Riemannian manifolds. It turns out that in addition to the possibilities listed above, a conformally flat pseudo-Riemannian manifold can have a weakly irreducible not irreducible holonomy group. We gave a complete local description of conformally flat Lorentzian manifolds (M,g) with weakly irreducible not irreducible holonomy groups.

On a Walker manifold (M,g) we define the canonical function λ by the equality

$$\mathrm{Ric}(p)=\lambda p,$$

where Ric is the Ricci operator. If the metric g is written in the form (4.11), then $\lambda=(1/2)\partial_v^2 H$, and the scalar curvature of g satisfies

$$s=2\lambda+s_0,$$

where s_0 is the scalar curvature of the metric h. The form of a conformally flat Walker metric will depend on the vanishing of the function λ. In the general case we obtain the following result.

Theorem 31. *Let (M,g) be a conformally flat Walker manifold (that is, the Weyl curvature tensor is equal to zero) of dimension $n+2\geqslant 4$. Then in a neighbourhood of each point of M there exist coordinates $v,x^1,\dots,x^n,u$ such that*

$$g=2\,dv\,du+\Psi\sum_{i=1}^n(dx^i)^2+2A\,du+\bigl(\lambda(u)v^2+vH_1+H_0\bigr)(du)^2,$$

where

$$\Psi = 4\Big(1-\lambda(u)\sum_{k=1}^{n}(x^k)^2\Big)^{-2},$$

$$A = A_i\,dx^i, \quad A_i = \Psi\Big(-4C_k(u)x^kx^i + 2C_i(u)\sum_{k=1}^{n}(x^k)^2\Big),$$

$$H_1 = -4C_k(u)x^k\sqrt{\Psi} - \partial_u\log\Psi + K(u),$$

$$H_0(x^1,\dots,x^n,u) = \begin{cases} \frac{4}{\lambda^2(u)}\Psi\sum_{k=1}^{n}C_k^2(u) \\ \quad + \sqrt{\Psi}\left(a(u)\sum_{k=1}^{n}(x^k)^2 + D_k(u)x^k + D(u)\right), & \text{if } \lambda(u)\neq 0, \\ 16\Big(\sum_{k=1}^{n}(x^k)^2\Big)^2\sum_{k=1}^{n}C_k^2(u) \\ \quad + \tilde{a}(u)\sum_{k=1}^{n}(x^k)^2 + \tilde{D}_k(u)x^k + \tilde{D}(u), & \text{if } \lambda(u)=0, \end{cases}$$

for some functions $\lambda(u)$, $a(u)$, $\tilde{a}(u)$, $C_i(u)$, $D_i(u)$, $D(u)$, $\tilde{D}_i(u)$, and $\tilde{D}(u)$.

The scalar curvature of the metric g is equal to $-(n-2)(n+1)\lambda(u)$.

If the function λ is zero on some open set or it is non-vanishing, then the above metric can be simplified.

Theorem 32. *Let (M,g) be a conformally flat Walker Lorentzian manifold of dimension $n+2\geqslant 4$.*

1) *If the function λ is non-zero at a point, then in a neighbourhood of this point there exist coordinates $v,x^1,\dots,x^n,u$ such that*

$$g = 2\,dv\,du + \Psi\sum_{i=1}^{n}(dx^i)^2 + \left(\lambda(u)v^2 + vH_1 + H_0\right)(du)^2,$$

where

$$\Psi = 4\Big(1-\lambda(u)\sum_{k=1}^{n}(x^k)^2\Big)^{-2},$$

$$H_1 = -\partial_u\log\Psi, \qquad H_0 = \sqrt{\Psi}\left(a(u)\sum_{k=1}^{n}(x^k)^2 + D_k(u)x^k + D(u)\right).$$

2) *If $\lambda\equiv 0$ in some neighbourhood of a point, then in a neighbourhood of this point there exist coordinates $v,x^1,\dots,x^n,u$ such that*

$$g = 2\,dv\,du + \sum_{i=1}^{n}(dx^i)^2 + 2A\,du + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i\,dx^i, \quad A_i = C_i(u)\sum_{k=1}^n (x^k)^2, \quad H_1 = -2C_k(u)x^k,$$

$$H_0 = \sum_{k=1}^n (x^k)^2\biggl(\frac{1}{4}\sum_{k=1}^n (x^k)^2\sum_{k=1}^n C_k^2(u) - (C_k(u)x^k)^2 + \dot{C}_k(u)x^k + a(u)\biggr) + D_k(u)x^k + D(u).$$

In particular, if all the C_i *are equal to* 0, *then the metric can be rewritten in the form*

$$g = 2\,dv\,du + \sum_{i=1}^n (dx^i)^2 + a(u)\sum_{k=1}^n (x^k)^2\,(du)^2. \tag{10.1}$$

Thus, Theorem 32 gives the local form of a conformally flat Walker metric in neighbourhoods of points where λ is non-zero or constantly zero. Such points form a dense subset of the manifold. Theorem 31 also describes the metric in neighbourhoods of points at which λ vanishes but is not locally zero, that is, in neighbourhoods of isolated zero points of λ.

We next find the holonomy algebras of the metrics obtained and check which of the metrics are decomposable.

Theorem 33. *Let* (M, g) *be as in Theorem* 31.

1) *The manifold* (M, g) *is locally indecomposable if and only if there exists a coordinate system with one of the following properties*:

- $\dot{\lambda} \not\equiv 0$;
- $\dot{\lambda} \equiv 0$, $\lambda \neq 0$, *that is*, g *can be written as in the first part of Theorem* 32, *and*
$$\sum_{k=1}^n D_k^2 + (a + \lambda D)^2 \not\equiv 0;$$
- $\lambda \equiv 0$, *that is*, g *can be written as in the second part of Theorem* 32, *and*
$$\sum_{k=1}^n C_k^2 + a^2 \not\equiv 0.$$

Otherwise, the metric can be written in the form

$$g = \Psi\sum_{k=1}^n (dx^k)^2 + 2\,dv\,du + \lambda v^2\,(du)^2, \qquad \lambda \in \mathbb{R}.$$

The holonomy algebra of this metric is trivial if and only if $\lambda = 0$. *If* $\lambda \neq 0$, *then the holonomy algebra is isomorphic to* $\mathfrak{so}(n) \oplus \mathfrak{so}(1,1)$.

2) *Suppose that the manifold* (M, g) *is locally indecomposable. Then its holonomy algebra is isomorphic to* $\mathbb{R}^n \subset \mathfrak{sim}(n)$ *if and only if*

$$\lambda^2 + \sum_{k=1}^n C_k^2 \equiv 0$$

for all coordinate systems. In this case (M,g) *is a pp-wave, and* g *is given by* (10.1). *If for each coordinate system*

$$\lambda^2+\sum_{k=1}^{n}C_k^2\not\equiv 0,$$

then the holonomy algebra is isomorphic to $\mathfrak{sim}(n)$.

Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds were classified in [81], where the problem was posed of constructing an example of a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (denoted there by R_{14}). An attempt at constructing such a metric was made in [75]. We show that the metric constructed there is in fact decomposable and its holonomy algebra is $\mathfrak{so}(1,1)\oplus\mathfrak{so}(2)$. Thus, in this paper we get conformally flat metrics with the holonomy algebra $\mathfrak{sim}(n)$ for the first time, and even more, we find all such metrics.

The field equations of Nordström's theory of gravitation, which was conceived before Einstein's theory, have the form

$$W=0,\qquad s=0$$

(see [106], [119]). All the metrics in Theorem 31 for dimension 4 together with the higher-dimensional metrics in the second part of Theorem 32 provide examples of solutions of these equations. Thus, we have found all solutions to Nordström's theory of gravity with holonomy algebras contained in $\mathfrak{sim}(n)$. Above we saw that a complete solution of the Einstein equation on Lorentzian manifolds with such holonomy algebras is lacking.

An important fact is that a simply connected conformally flat Lorentzian spin manifold admits the space of conformal Killing spinors of maximal dimension [12].

It would be interesting to obtain examples of conformally flat Lorentzian manifolds with certain global geometric properties. For example, constructions of globally hyperbolic Lorentzian manifolds with special holonomy groups are important [17], [19].

The projective equivalence of 4-dimensional conformally flat Lorentzian metrics with special holonomy algebras was recently studied in [80]. There are many interesting papers about conformally flat (pseudo-)Riemannian manifolds, in particular, Lorentzian manifolds. We mention some of them: [6], [84], [93], [115].

11. 2-symmetric Lorentzian manifolds

In this section we discuss the classification obtained in [5] for 2-symmetric Lorentzian manifolds.

Symmetric pseudo-Riemannian manifolds form an important class of spaces. A direct generalization of these manifolds is provided by the so-called k-symmetric pseudo-Riemannian spaces (M,g), which satisfy the conditions

$$\nabla^k R=0,\quad \nabla^{k-1}R\neq 0,$$

where $k\geqslant 1$. In the case of Riemannian manifolds, the condition $\nabla^k R=0$ implies that $\nabla R=0$ [117]. On the other hand, there exist pseudo-Riemannian k-symmetric spaces for $k\geqslant 2$ [28], [90], [112].

The list of indecomposable simply connected Lorentzian symmetric spaces is exhausted by the de Sitter spaces, the anti-de Sitter spaces, and the Cahen–Wallach spaces, which are special pp-waves. Kaigorodov [90] considered different generalizations of Lorentzian symmetric spaces.

Senovilla's paper [112] is the start of a systematic investigation of 2-symmetric Lorentzian spaces. There it was proved that any 2-symmetric Lorentzian space admits a parallel isotropic vector field. In [28] a local classification of 4-dimensional 2-symmetric Lorentzian spaces was obtained with the use of Petrov's classification of the Weyl tensors [108].

In [5] we generalized the result in [28] to the case of arbitrary dimension.

Theorem 34. *Let (M,g) be a locally indecomposable Lorentzian manifold of dimension $n+2$. Then (M,g) is 2-symmetric if and only if locally there are coordinates $v,x^1,\dots,x^n,u$ such that*

$$g = 2\,dv\,du + \sum_{i=1}^{n}(dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j\,(du)^2,$$

where H_{ij} is a non-zero diagonal real matrix with the diagonal elements $\lambda_1 \leqslant \dots \leqslant \lambda_n$, and F_{ij} is a symmetric real matrix.

From the Wu theorem it follows that any 2-symmetric Lorentzian manifold is locally a product of an indecomposable 2-symmetric Lorentzian manifold and a locally symmetric Riemannian manifold. In [29] it was shown that a simply connected geodesically complete 2-symmetric Lorentzian manifold is the product of $\mathbb{R}^{n+2}$ with the metric in Theorem 34 and a (possibly trivial) Riemannian symmetric space.

The proof of Theorem 34 given in [5] demonstrates the methods of the theory of holonomy groups in the best way. Let $\mathfrak{g} \subset \mathfrak{so}(1,n+1)$ be the holonomy algebra of the manifold (M,g). Consider the space $\mathscr{R}^\nabla(\mathfrak{g})$ of covariant derivatives of the algebraic curvature tensors of type $\mathfrak{g}$ which consists of the linear maps from $\mathbb{R}^{1,n+1}$ to $\mathscr{R}(\mathfrak{g})$ that satisfy the second Bianchi identity. Let $\mathscr{R}^\nabla(\mathfrak{g})_\mathfrak{g} \subset \mathscr{R}^\nabla(\mathfrak{g})$ be the subspace annihilated by the algebra $\mathfrak{g}$.

The tensor ∇R is parallel and non-zero, hence its value at each point of the manifold belongs to the space $\mathscr{R}^\nabla(\mathfrak{g})_\mathfrak{g}$. The space $\mathscr{R}^\nabla(\mathfrak{so}(1,n+1))_\mathfrak{g}$ is trivial [116], therefore $\mathfrak{g} \subset \mathfrak{sim}(n)$.

The cornerstone of the proof is the equality $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{sim}(n)$, that is, $\mathfrak{g}$ is the algebra of type 2 with trivial orthogonal part $\mathfrak{h}$. Such a manifold is a pp-wave (see § 4.2), that is, locally,

$$g = 2\,dv\,du + \sum_{i=1}^{n}(dx^i)^2 + H\,(du)^2, \qquad \partial_v H = 0,$$

and the equation $\nabla^2 R = 0$ can easily be solved.

Suppose that the orthogonal part $\mathfrak{h} \subset \mathfrak{so}(n)$ of the holonomy algebra $\mathfrak{g}$ is non-trivial. The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ can be decomposed into irreducible parts, as in § 4.1. Using the coordinates (4.12) allows us to assume that the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible. If $\mathfrak{g}$ is of type 1 or 3, then simple algebraic computations show that $\mathscr{R}^\nabla(\mathfrak{g})_\mathfrak{g} = 0$.

We are left with the case $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$, where the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible. In this case the space $\mathscr{R}^{\nabla}(\mathfrak{g})_{\mathfrak{g}}$ is 1-dimensional, which enables us to find the explicit form of the tensor ∇R, namely, if the metric g has the form (4.11), then

$$\nabla R = f\, du \otimes h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j)$$

for some function f.

With the use of the last equality it was proved that $\nabla W = 0$, that is, the Weyl conformal tensor W is parallel. The results in [49] show that $\nabla R = 0$, or $W = 0$, or the manifold under consideration is a pp-wave. The first condition contradicts the assumption that $\nabla R \neq 0$, and the last condition contradicts the assumption that $\mathfrak{h} \neq 0$. From the results in § 10 it follows that the condition $W = 0$ implies the equality $\mathfrak{h} = 0$, so that again we get a contradiction.

It turns out that the last step of the proof in [5] can be appreciably simplified and it is not necessary to consider the condition $\nabla W = 0$. Indeed, let us return to the equality for ∇R. It is easy to check that $\nabla du = 0$. Therefore, the equality $\nabla^2 R = 0$ implies that $\nabla(f h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j)) = 0$. Consequently,

$$\nabla(R - uf h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j)) = 0.$$

The value of the tensor field $R - uf h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j)$ at each point of the manifold belongs to the space $\mathscr{R}(\mathfrak{g})$ and is annihilated by the holonomy algebra $\mathfrak{g}$. This immediately implies that

$$R - uf h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j) = f_0 h^{ij}(p \wedge \partial_i) \otimes (p \wedge \partial_j)$$

for some function f_0, that is, R is the curvature tensor of a pp-wave, which contradicts the condition $\mathfrak{h} \neq 0$. Thus, $\mathfrak{h} = 0$, and $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{sim}(n)$.

Theorem 34 was re-proved in [29] by considering the equation $\nabla^2 R = 0$ in local coordinates and going through extensive computations. This shows the significant advantage of holonomy group methods.

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Received 16/JUL/14
Translated by THE AUTHOR